

Algebraic and Lagrangian Subvarieties on Mirror Pairs

Yong Seung Cho

Department of Mathematics, Ewha Womans University, Seoul, Republic of Korea

Email: yescho@ewha.ac.kr

How to cite this paper: Cho, Y.S. (2026) Algebraic and Lagrangian Subvarieties on Mirror Pairs. *Advances in Pure Mathematics*, 16, 815-825.

<https://doi.org/10.4236/apm.2026.1610040>

Received: April 23, 2026

Accepted: October 19, 2026

Published: October 22, 2026

Copyright © 2026 by author(s) and Scientific Research Publishing Inc.

This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).

<http://creativecommons.org/licenses/by/4.0/>



Open Access

Abstract

On a mirror pair of Calabi-Yau n -folds, there is a one-to-one correspondence between the elements of the Hodge group and those of the n -dimensional cohomology group of its mirror space. In this paper, we show that there is an isomorphism between the group of the algebraic cycles and the group of the Lagrangian subvarieties of the mirror spaces. This correspondence provides a geometric realization of mirror symmetry at the level of subvarieties.

Keywords

Calabi-Yau Manifold, Holonomy, Superconformal Field Theory, Mirror Pair, Algebraic Variety, Lagrangian Variety, Quintic Threefold

1. Introduction

Let (M, ω) and (M^0, ω^0) be a mirror pair of Calabi-Yau n -folds. Then, using the properties of Calabi-Yau manifolds and the mirror map, we have a duality between the Hodge classes of M and the n -dimensional cohomology classes of M^0 , and vice versa [1]-[3]. In [1], we showed that, for Hodge manifolds, the Hodge groups coincide with the corresponding algebraic groups. In this paper, we show that there is a one-to-one correspondence between the algebraic subvarieties of M and the Lagrangian subvarieties of M^0 . This correspondence may be regarded as a geometric manifestation of mirror symmetry at the level of subvarieties.

Throughout this paper, the dimensions of manifolds are complex dimensions, whereas those of Lagrangians are real dimensions, equal to half the real dimensions of the ambient manifolds.

In Section 2, we recall the notion of Hodge manifolds, Kodaira's projective embedding theorem [4], Lagrangian subvarieties, and the fact that Hodge classes are

algebraic. In Section 3, we review Calabi-Yau varieties, superconformal field theory, and mirror symmetry [5] [6]. In Section 4, we introduce algebraic and Lagrangian subvarieties and discuss their relationships for a mirror pair of Calabi-Yau varieties [7]-[11]. In Section 5, we examine the standard mirror pair of quintic 3-folds [12]. We often use the terms “variety” and “n-fold” interchangeably with “manifold”, since we allow mild singularities such as orbifolds.

2. Hodge Manifolds

A Hodge manifold [1] [2] is a compact Kähler manifold (M, h, ω, J) whose fundamental 2-form ω is integrable. Note in Kähler manifolds, the 2-form ω is closed and $[\omega] \in H^2(M, \mathbb{R})$, whereas in Hodge manifolds $[\omega] \in H^2(M, \mathbb{Z})$.

2.1. Algebraic Subvarieties

2.1.1. Kodaira's Projective Embedding Theorem [4]

A compact Kähler manifold (M, h, ω, J) is Hodge if and only if M is holomorphically embedded in a projective space $\mathbb{P}^N$ for some N . $\square$

Let M be a Hodge n -fold. Then the Hodge decompositions are for $0 \leq k \leq 2n$, $0 \leq p, q \leq n$,

$$H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M), \quad \overline{H^{p,q}(M)} \cong H^{q,p}(M)$$

where $H^{p,q}(M)$ is the complex subspace of $H^k(M, \mathbb{C})$ of type (p, q) classes. In particular, $H^n(M, \mathbb{C}) = H^{n,0}(M) \oplus H^{n-1,1}(M) \oplus \cdots \oplus H^{0,n}(M)$.

Let $\text{Hdg}^{2p}(M) := H^{p,p}(M) \cap H^{2p}(M, \mathbb{Q})$ be the space of Hodge classes of M of degree $2p$, and $\text{Hdg}^*(M) := \bigoplus_{p=0}^n \text{Hdg}^{2p}(M)$ which is called the spaces of all Hodge classes of M . Let $H^{2p}(M, \mathbb{Q})_{\text{alg}}$ be the space of algebraic subvarieties of codimension p in M with rational coefficients. Then we had the following 2.1.2 Theorem, for the proof see section 4.9, Theorem 1 in [1].

2.1.2. Theorem [1]

Let M be a Hodge manifold of dimension n . Then for each $0 \leq p \leq n$, there is an isomorphism

$$\text{Hdg}^{2p}(M) \xrightarrow{\sim} H^{2p}(M, \mathbb{Q})_{\text{alg}}$$

and consequently

$$\text{Hdg}^*(M) = \bigoplus_{p=0}^n \text{Hdg}^{2p}(M) \xrightarrow{\sim} \bigoplus_{p=0}^n H^{2p}(M, \mathbb{Q})_{\text{alg}} = H^*(M, \mathbb{Q})_{\text{alg}}.$$

$\square$

2.2. Lagrangian Subvarieties

Let (M, ω, J) be a Hodge manifold of dimension n . A Lagrangian subvariety of M is a subvariety L of M which is a maximal isotropic subvariety, hence L is a subvariety on which the Hodge 2-form ω vanishes which is maximal with this property. Thus, if a subvariety $L \subset M$ is Lagrangian, then $\dim M = 2 \dim L$ and the restriction $\omega|_{TL} = 0$.

Examples

- 1) Every real curve in a Riemann surface is Lagrangian.
- 2) Every smooth manifold is Lagrangian in its cotangent bundle with the canonical symplectic form.
- 3) The product of Lagrangian subspaces is Lagrangian in the product of the spaces containing the Lagrangian subspaces.

2.3. The Group of Lagrangian Classes

Let (M, ω, J) be a Hodge manifold of dimension n . The Hodge decomposition of the middle dimensional cohomology of M is

$$H^n(M, \mathbb{C}) = \bigoplus_{p=0}^n H^{n-p,p}(M), \quad \overline{H^{n-p,p}(M)} \cong H^{p,n-p}(M)$$

and the Poincaré duality $PD = \cap[M]: H^n(M, \mathbb{C}) \xrightarrow{\sim} H_n(M, \mathbb{C})$ is an isomorphism.

Notations

Define $PD(H^{n-p,p}(M)) := H_{p,n-p}(M)$, $Hlg^{n-p,p}(M) := H^{n-p,p}(M) \cap H^n(M, \mathbb{Q})$, $Hlg_{p,n-p}(M) := H_{p,n-p}(M) \cap H_n(M, \mathbb{Q})$, and

$$Hlg^*(M) := \bigoplus_{p=0}^n Hlg^{n-p,p}(M), \quad Hlg_*(M) := \bigoplus_{p=0}^n Hlg_{p,n-p}(M),$$

which are called the Lagrangian cohomology and homology groups of Lagrangian classes of M , respectively.

3. Calabi-Yau Manifolds

Eugenio Calabi conjectured that any compact Kähler manifold with vanishing the first Chern class admits a Kähler metric with vanishing Ricci tensor. The conjecture was proved by Shing-Tung Yau. After that the manifolds are called Calabi-Yau.

3.1. The Calabi-Yau n -Folds

A Calabi-Yau n -fold is a Hodge manifold (M, ω) of dimension n satisfying one of following conditions, in this paper we use a Hodge manifold instead of a compact Kähler manifold:

- 1) The canonical line bundle of M is trivial, $K_M \simeq \mathcal{O}_M$.
- 2) M has a holomorphic n -form Ω that vanishes nowhere.
- 3) The structure group of the tangent bundle TM is $SU(n)$.
- 4) M has a Hodge metric with global holonomy contained in $SU(n)$.

Remark

Let we have (M, ω) be a Calabi-Yau n -fold. Then we have the Hodge class $[\omega] \in H^2(M, \mathbb{Z})$, $c_1(TM) = 0$, and Hodge numbers $h^{1,0} = \dots = h^{n-1,0} = 0$, and nonzero holomorphic n -form class $[\Omega] \in H^{n,0}(M) = \mathbb{C}([\Omega]) \simeq \mathbb{C}$.

The holomorphic n -form Ω is locally decomposable

$$\Omega = \Omega_1 + i\Omega_2 \quad \text{and} \quad d\Omega = d\Omega_1 = d\Omega_2 = 0$$

$\Omega_1 \wedge \omega = \Omega_2 \wedge \omega = 0$, and the $2n$ -form

$$(\Omega_1 + i\Omega_2) \wedge (\Omega_1 - i\Omega_2) = \begin{cases} \omega^n & \text{if } n \text{ is even,} \\ i\omega^n & \text{if } n \text{ is odd.} \end{cases}$$

3.2. Holonomy on Threefolds

In physics the space time should be a 10-dimensional space [3] [5] [6]. The space is locally a product $S^4 \times M^6$, where S^4 is the usual space-time and M is a compact 6-dimensional Riemannian manifold which is too small to be seen at macroscopic scale but is crucial for quantization. For the supersymmetry and consistency requirements we impose a condition on the metric of M that the holonomy group of the metric is $SU(3)$.

Then the M has the following properties:

- 1) The metric on M is Ricci flat and $b_1(M) = 0$,
- 2) There is a complex structure on M such that $c_1(TM) = 0$ and the metric is Kähler,
- 3) M is a Calabi-Yau 3-fold,
- 4) M has the good properties, such as supersymmetry, conformality and quantization called the superconformal field theory (SCFT).

3.3. Mirror Symmetry

Let (M, ω) be a Calabi-Yau n -fold. Then on M the associated superconformal algebra contains two $\mathcal{U}(1)$ subalgebras, one comes from infinitesimal rotation of the two supersymmetries in the left and the other in the right moving sector.

3.3.1. Representations of $\mathcal{U}(1) \times \mathcal{U}(1)$

The subalgebra $\mathcal{U}(1) \times \mathcal{U}(1)$ has a generator (P, Q) . However, there is no canonical choice of generator for each $\mathcal{U}(1)$. If we change the order of the supersymmetries, then the sign of the generator is changed. The generator (P, Q) of $\mathcal{U}(1) \times \mathcal{U}(1)$ can be regarded as operators on the various Hilbert spaces of states. Then the Hilbert space decomposes into eigenspaces under the action of (P, Q) .

The $\mathcal{U}(1) \times \mathcal{U}(1)$ representation on the Hilbert spaces are:

$$(p, q)\text{-eigenspace} : H^q(M, \wedge^p TM), \text{ and}$$

$$(-p, q)\text{-eigenspace} : H^q(M, \Omega^p M).$$

The most interesting eigenspaces are the $(1, 1)$ -eigenspace $H^1(M, TM)$ and the $(-1, 1)$ -eigenspace $H^1(M, \Omega^1 M)$. Because $H^1(M, TM)$ is a tangent space of the moduli space $\mathcal{C}_{\mathfrak{M}}(M)$ of the complex structures of M , and $H^1(M, \Omega^1 M)$ is the tangent space of the moduli space $\mathcal{K}_{\mathfrak{M}}(M)$ of the complexified Kähler structures of M .

3.3.2. Mirror Pairs

Let the generator P change to $-P$ and Q leave to Q . Then the (p, q) -eigenspace $H^q(M, \wedge^p TM)$ is changed to the $(-p, q)$ -eigenspace $H^q(M, \Omega^p M)$.

However, in general they are not isomorphic. If there is a Calabi-Yau variety (M^0, ω^0) such that $H^q(M, \Omega^p M) \simeq H^q(M^0, \wedge^p TM^0)$ and $H^q(M, \wedge^p TM) \simeq H^q(M^0, \Omega^p M^0)$ for all p, q , then the Hodge diamonds of (M, ω) and (M^0, ω^0) are symmetric and they are called a mirror pair.

Let (M, ω) and (M^0, ω^0) be a mirror pair of Calabi-Yau n -folds. Then for all p and q ,

$$H^q(M, \Omega^p M) \simeq H^q(M^0, \wedge^p TM^0) \simeq H^q(M^0, \Omega^{n-p} M^0),$$

since $\Omega^n(M^0) \simeq \mathcal{O}_{M^0}$.

If $p = q$, then $H^{p,p}(M) \simeq H^{n-p,p}(M^0)$, and

$$H^{\text{mid}}(M, \mathbb{C}) := \bigoplus_{p=0}^n H^{p,p}(M) \simeq \bigoplus_{p=0}^n H^{n-p,p}(M^0) = H^n(M^0, \mathbb{C}).$$

Similarly, we have $H^{n-p,p}(M) \simeq H^{p,p}(M^0)$ and $H^n(M, \mathbb{C}) \simeq H^{\text{mid}}(M^0, \mathbb{C})$.

Thus we have:

Theorem

Let (M, ω) and (M^0, ω^0) be a mirror pair of Calabi-Yau n -folds. Then we have the following isomorphisms;

$$H^{\text{mid}}(M, \mathbb{C}) \simeq H^n(M^0, \mathbb{C}) \text{ and } H^n(M, \mathbb{C}) \simeq H^{\text{mid}}(M^0, \mathbb{C}).$$

□

4. Algebraic and Lagrangian Cycles

Let (M, ω) and (M^0, ω^0) be a mirror pair of Calabi-Yau n -folds. Then for $0 \leq p \leq n$, by Theorem 2.1.2, the following pairs of groups

$$\text{Hdg}^{2p}(M) \text{ and } H^{2p}(M, \mathbb{Q})_{\text{alg}}, \text{Hdg}^{2p}(M^0) \text{ and } H^{2p}(M^0, \mathbb{Q})_{\text{alg}},$$

$$\text{Hdg}^*(M) \text{ and } H^*(M, \mathbb{Q})_{\text{alg}}, \text{Hdg}^*(M^0) \text{ and } H^*(M^0, \mathbb{Q})_{\text{alg}}$$

are isomorphic. Also by Theorem in 3.3.2,

$$H^{\text{mid}}(M, \mathbb{C}) \text{ and } H^n(M^0, \mathbb{C}), H^n(M, \mathbb{C}) \text{ and } H^{\text{mid}}(M^0, \mathbb{C})$$

are isomorphic.

As the Hodge groups, we define the groups of Lagrangian classes by

$$\text{Hlg}^{n-p,p}(M) := H^{n-p,p}(M) \cap H^n(M, \mathbb{Q}),$$

$$\text{Hlg}^{n-p,p}(M^0) := H^{n-p,p}(M^0) \cap H^n(M^0, \mathbb{Q}),$$

and

$$\text{Hlg}(M) := \bigoplus_{p=0}^n \text{Hlg}^{n-p,p}(M), \text{Hlg}(M^0) := \bigoplus_{p=0}^n \text{Hlg}^{n-p,p}(M^0).$$

Also we define the groups of Lagrangian subvarieties by

$$H^{n-p,p}(M, \mathbb{Q})_{\text{lag}}$$

$$:= \left\{ \sum \ell_i L_i \mid \ell_i \in \mathbb{Q}, L_i : \text{a } (n-p, p)\text{-type Lagrangian subvariety in } M \right\}.$$

A Lagrangian subvariety which belongs to $H^{n-p,p}(M, \mathbb{Q})_{\text{lag}}$ is called a $(n-p, p)$ -type Lagrangian subvariety, which is locally spanned by

$\{\bar{z}_{i_1}, \dots, \bar{z}_{i_p}, z_{j_1}, \dots, z_{j_{n-p}}\}$ with real coefficients.

By the Jacobian transformation gluing pieces into complex manifold, a kind of sheaves, we can glue the algebraic and Lagrangian pieces into global algebraic and Lagrangian sheaves, respectively [9] [11]. And we define

$$H^n(M, \mathbb{Q})_{\text{lag}} := \bigoplus_{p=0}^n H^{n-p,p}(M, \mathbb{Q})_{\text{lag}}.$$

Similarly, we define $H^{n-p,p}(M^0, \mathbb{Q})_{\text{lag}}$ and $H^n(M^0, \mathbb{Q})_{\text{lag}}$.

4.1. Hodge Classes and Algebraic Cycles

Let (M, ω) be a Hodge manifold of dimension n . A Hodge class $[\alpha] \in \text{Hdg}^{2p}(M)$ can be locally written by

$$\alpha = \sum_{|I|=p} \alpha_I dz_I \wedge d\bar{z}_I, \quad \alpha_I \in \mathbb{Q}, \quad I = \{i_1, \dots, i_p\}.$$

Let the Poincaré dual

$$\text{PD} = \cap[M]: H^{2p}(M, \mathbb{C}) \cap H^{2p}(M, \mathbb{Q}) \rightarrow H_{2n-2p}(M, \mathbb{Q}) \cap H_{2n-2p}(M, \mathbb{Q})$$

have the image $\text{PD}(\text{Hdg}^{2p}(M)) = H_{n-p, n-p}(M) \cap H_{2n-2p}(M, \mathbb{Q}) := \text{Hdg}_{2n-2p}(M)$, and $A = \text{PD}([\alpha]) = [\alpha] \cap [M]$. Then the A can be locally written by

$$A = \sum_{|I|=p} \alpha'_I Z_J, \quad J = \{1, \dots, n\} - I$$

where the α'_I are determined by $[\alpha] \cap [M]$. By the gluing argument of local pieces, we have the Z_J , which is the subvariety locally spanned by $\{z_{j_1}, \dots, z_{j_{n-p}}\}$ and $z_{i_1}, \dots, z_{i_p}$ constants. Thus as a homology $A \in H_{2n-2p}(M, \mathbb{Q})_{\text{alg}}$ and as a cohomology $[A] = [\alpha] \in H^{2p}(M, \mathbb{Q})_{\text{alg}}$ is an algebraic cycle in M .

Theorem

Let M be a Hodge manifold of dimension n . If a Hodge class $[\alpha] \in \text{Hdg}^{2p}(M)$ has a local form $\alpha = \sum_{|I|=p} \alpha_I dz_I \wedge d\bar{z}_I$, then we have

$$\text{PD}([\alpha]) = A (= \sum \alpha'_I Z_J : \text{local}) \in H_{2n-2p}(M, \mathbb{Q})_{\text{alg}}$$

and as a cohomology class

$$[\alpha] = [A] \in H^{2p}(M, \mathbb{Q})_{\text{alg}}.$$

□

Note. Theorem 4.1.1 is another version in Theorem 2.1.2.

4.2. Lagrangian Classes and Lagrangian Cycles

Let (M, ω, J) be a Hodge manifold of dimension n . A Lagrangian class $[\beta] \in \text{Hlg}^{n-p,p}(M) = H^{n-p,p}(M) \cap H^n(M, \mathbb{Q})$ can be locally written by a form:

$$\beta = \sum_{|I|=p} \beta_I dz_J \wedge d\bar{z}_I$$

where $J = \{1, \dots, n\} - I$, $I = \{i_1, \dots, i_p\}$.

Let $\text{PD}: \text{Hlg}^{n-p,p}(M) \rightarrow \text{Hlg}_{p, n-p}(M)$ be the restriction of Poincaré duality.

For a Lagrangian class $[\beta] \in \text{Hlg}^{n-p,p}(M)$, the image

$\text{PD}([\beta]) = [\beta] \cap [M] := L_\beta$ can be locally written by:

$$\left\langle x_{i_1} - iy_{i_1}, \dots, x_{i_p} - iy_{i_p}, x_{j_1} + iy_{j_1}, \dots, x_{j_{n-p}} + iy_{j_{n-p}} \right\rangle$$

which is the space generated by the elements with real coefficients in the space generated by $\left\langle \bar{z}_{i_1}, \dots, \bar{z}_{i_p}, z_{j_1}, \dots, z_{j_{n-p}} \right\rangle$ with complex coefficients. Then the space L_β has a real n -dimension. Since $\omega = \frac{i}{2} \sum_j dz_j \wedge d\bar{z}_j$ and the tangent space TL_β is generated by:

$$\partial_j = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \bar{\partial}_k = \frac{1}{2} \left(\frac{\partial}{\partial x_k} + i \frac{\partial}{\partial y_k} \right)$$

where $j = i_1, \dots, i_p$ and $k = j_1, \dots, j_{n-p}$, the $\omega = 0$ on TL_β and $TL_\beta \oplus J(TL_\beta) = TM|_{L_\beta}$. By the analytic continuation we have that the L_β is a Lagrangian subspace in M .

The submanifold L_β corresponding to a Lagrangian class $[\beta] \in \text{Hlg}^{n-p,p}(M)$ is a $(n-p, p)$ -type Lagrangian subvariety of M . Then we have:

Theorem

Let (M, ω, J) be a compact Kähler manifold of dimension n . Then for each Lagrangian class $[\beta] \in \text{Hlg}^{n-p,p}(M)$, the image $\text{PD}([\beta]) := L_\beta \subset M$ is represented as a Lagrangian subspace of M and as a cohomology cycle

$$[L_\beta] = \text{PD}^{-1} \text{PD}([\beta]) = [\beta] \in H^n(M, \mathbb{Q})_{\text{lag}}, \text{ that is,}$$

$$\text{Hlg}^{n-p,p}(M) \subset H^n(M, \mathbb{Q})_{\text{lag}}, \quad \text{Hlg}^n(M) \subset H^n(M, \mathbb{Q})_{\text{lag}},$$

and clearly $H^n(M, \mathbb{Q})_{\text{lag}} \subset \text{Hlg}^n(M)$. Thus, we have $\text{Hlg}^n(M) = H^n(M, \mathbb{Q})_{\text{lag}}$. $\square$

4.3. Algebraic and Lagrangian Subvarieties

If Calabi-Yau manifolds (M, ω, J) and (M^0, ω^0, J^0) are a mirror pair, then they have isomorphic superconformal field theories, especially the Kähler moduli space of M and the complex moduli space of M^0 are isomorphic and vice versa, also they have the isomorphism of 4.3.1. The isomorphisms are collectively called the mirror map of mirror pair.

Let (M, ω, J) and (M^0, ω^0, J^0) be a mirror pair of Calabi-Yau n -folds.

4.3.1. Mirror Maps

The mirror maps, for $0 \leq p \leq n$,

$$H^{p,p}(M) \xrightarrow{\sim} H^{n-p,p}(M^0), \quad H^{n-p,p}(M) \xrightarrow{\sim} H^{p,p}(M)$$

and

$$H^{\text{mid}}(M) \xrightarrow{\sim} H^n(M^0, \mathbb{C}), \quad H^n(M, \mathbb{C}) \xrightarrow{\sim} H^{\text{mid}}(M^0)$$

are isomorphic. The mirror maps induce the following commutative diagram of isomorphisms: for $0 \leq p \leq n$,

$$\begin{array}{ccc}
 \mathrm{Hdg}^{2p}(M) & \xrightarrow{m^p} & H^{n-p,p}(M^0)_{\mathrm{lag}} \\
 \downarrow \cap[M] & & \downarrow \cap[M^0] \\
 \mathrm{Hdg}_{2n-2p}(M) & \xrightarrow{m_p} & H_{p,n-p}(M^0)_{\mathrm{lag}}
 \end{array}
 \quad (*)$$

where $\cap[M]$ and $\cap[M^0]$ are the Poincaré duals.

4.3.2. Local Descriptions

Let $\{z_1, \dots, z_n\}$ and $\{w_1, \dots, w_n\}$ be local holomorphic coordinates of M and M^0 , respectively, and $z_j = x_j + iy_j$, $w_j = s_j + it_j$ for $j = 1, \dots, n$. Let $[\alpha] \in \mathrm{Hdg}^{2p}(M)$ be a Hodge class with local form

$$\alpha = \sum_{|I|=p} \alpha_I dz_I \wedge d\bar{z}_I \in H^{p,p}(M), \quad \alpha_I \in \mathbb{Q}.$$

Then the image $m^p([\alpha]) := [\beta]$ has a local form:

$$\beta = \sum_{|J|=p} \beta_J dw_J \wedge d\bar{w}_J, \quad J = \{1, \dots, n\} - I.$$

Mirror Maps

The mirror map is defined by the composition:

$$\begin{aligned}
 m^p : \mathrm{Hdg}^{2p}(M) &= H^{p,p}(M) \cap H^{2p}(M, \mathbb{Q}) \rightarrow H^p(M^0, \Lambda^p TM^0) \cap H^n(M^0, \mathbb{Q}) \\
 &\rightarrow H^p(M^0, \Omega^{n-p} M^0) \cap H^n(M^0, \mathbb{Q}) = H^{n-p,p}(M^0)_{\mathrm{lag}},
 \end{aligned}$$

$$\begin{aligned}
 \alpha &= \sum_{|I|=p} \alpha_I dz_I \wedge d\bar{z}_I \\
 &\mapsto \sum_{|I|=p} \beta_I \partial w_I \wedge d\bar{w}_I \\
 &\mapsto \sum_{|I|=p} \beta_I \Omega^0 \wedge \partial w_I \wedge d\bar{w}_I \\
 &= \sum_{|I|=p} \beta_I dw_J \wedge d\bar{w}_J
 \end{aligned}$$

where $\Omega^0 = dw_1 \wedge \dots \wedge dw_n$ is a nowhere vanishing holomorphic n -form on M^0 and M^0 is Calabi-Yau, and $J = \{1, \dots, n\} - I$.

While the Poincaré dual $A := [\alpha] \cap [M] \in H_{2n-2p}(M)$ is represented by a codimension p subvariety of M with local coordinates $\{z_{j_1}, \dots, z_{j_{n-p}}\}$ where $\{j_1, \dots, j_{n-p}\} = \{1, \dots, n\} - I$.

As a cohomology class $[A] = [\alpha] \in H^{2p}(M, \mathbb{Q})$.

Lagrangians

Let $m^p : H^{2p}(M, \mathbb{Q})_{\mathrm{alg}} \rightarrow H^{n-p,p}(M^0, \mathbb{Q})_{\mathrm{lag}}$ be the mirror map and for $[\alpha] \in H^{2p}(M, \mathbb{Q})_{\mathrm{alg}}$, $[\beta] := m^p([\alpha])$. Then $[\beta]$ is locally represented by:

$$\beta = \sum_{|J|=p} \beta_J dw_J \wedge d\bar{w}_J, \quad J = \{1, \dots, n\} - I.$$

The Poincaré dual $B := [\beta] \cap [M^0]$ is a subvariety of M^0 which is locally represented by $(s_{i_1} - it_{i_1}, \dots, s_{i_p} - it_{i_p}, s_{j_1} + it_{j_1}, \dots, s_{j_{n-p}} + it_{j_{n-p}})$ with real coefficients.

The tangent space of B is the space spanned by:

$$\left\{ \frac{\partial}{\partial s_{i_1}} - i \frac{\partial}{\partial t_{i_1}}, \dots, \frac{\partial}{\partial s_{i_p}} - i \frac{\partial}{\partial t_{i_p}}, \frac{\partial}{\partial s_{j_1}} + i \frac{\partial}{\partial t_{j_1}}, \dots, \frac{\partial}{\partial s_{j_{n-p}}} + i \frac{\partial}{\partial t_{j_{n-p}}} \right\} \simeq \mathbb{R}^n \subset \mathbb{C}^n.$$

Since the fundamental 2-form on M^0 is

$$\omega^0 = \frac{i}{2} \sum_j dw_j \wedge d\bar{w}_j,$$

clearly we have $\omega^0 \equiv 0$ on B . Thus B is Lagrangian in M^0 and $[B] \in H_{p, n-p}(M^0)_{\text{lag}}$.

Theorem

Let (M, ω, J) and (M^0, ω^0, J^0) be a mirror pair of Calabi-Yau n -folds. Then the mirror maps induce the following commutative diagram of isomorphisms: for $0 \leq p \leq n$,

$$\begin{array}{ccc}
 H^{2p}(M, \mathbb{Q})_{\text{alg}} & \xrightarrow{m^p} & H^{n-p, p}(M^0, \mathbb{Q})_{\text{lag}} \\
 \downarrow \cap[M] & & \downarrow \cap[M^0] \\
 H_{2n-2p}(M, \mathbb{Q})_{\text{alg}} & \xrightarrow{m_p} & H_{p, n-p}(M^0, \mathbb{Q})_{\text{lag}}
 \end{array}
 \quad (**)$$

where $\cap[M]$ and $\cap[M^0]$ are the Poincaré duals. Similarly, there is a one-to-one correspondence between the Lagrangian subvarieties of M and the algebraic subvarieties of M^0 . $\square$

In theorem 4.3.2, $H^{2p}(M, \mathbb{Q})_{\text{alg}} = \text{Hdg}^{2p}(M) = H^{p, q}(M, \mathbb{Q})$, the m_p comes from mirror symmetry, and we define the map m_p by the composition

$$m_p = (\cap[M^0]) \circ m^p \circ (\cap[M])^{-1}$$

of isomorphisms.

4.4. Remark

4.4.1. Hodge and Algebraic Classes

On the Hodge manifolds the equality of Hodge classes and algebraic cycles comes from the Lefschetz decomposition, and the one of Lagrangian classes and Lagrangian varieties comes from the Hodge decomposition. The one-to-one correspondence comes from the mirror pair of Calabi-Yau n -folds.

4.4.2. Hodge and Algebraic Cycles

The diagram (*) in (4.3.1) is algebraic, and (**) of (4.3.2.3) is a geometric correspondence.

4.4.3. Homological Mirror Symmetry

In 1994, Kontsevich proposed a homological version of mirror symmetry: A pair (M, ω) and (M^0, ω^0) is homological mirror symmetric if and only if the derived category of coherent sheaves on (M^0, ω^0) is equivalent to the derived category of Lagrangian submanifolds of (M, ω) [10]. 4.3.2 Theorem implies the homological mirror symmetry.

5. Example: The Quintic Threefold

Let (M, ω, J) be the quintic threefold with its mirror (M^0, ω^0, J^0) which was constructed by Greene and Plesser [12]. Then Theorem 2.1.2 implies that the space of Hodge classes coincides with the space of algebraic cycles:

$$\text{Hdg}^*(M) = H^*(M)_{\text{alg}} = \langle M \rangle \oplus \langle H \rangle \oplus \langle H^2 \rangle \oplus \langle H^3 \rangle = \mathbb{Q}^4,$$

where $H = M \cap \mathbb{P}^3 \subset \mathbb{P}^4$ is a hypersurface of M , $M := H^0$, $H \in H^2(M)_{\text{alg}}$, $H^2 \in H^4(M)_{\text{alg}}$ and $H^3 \in H^6(M)_{\text{alg}}$. In fact, H^3 represents the class of a point.

While the Lagrangian group is $H^3(M^0, \mathbb{C})_{\text{lag}}$:

$$\begin{aligned} H^3(M^0, \mathbb{C})_{\text{lag}} &= H^{3,0}(M^0)_{\text{lag}} \oplus H^{2,1}(M^0)_{\text{lag}} \oplus H^{1,2}(M^0)_{\text{lag}} \oplus H^{0,3}(M^0)_{\text{lag}} \\ &= \langle L_0 \rangle \oplus \langle L_1 \rangle \oplus \langle L_2 \rangle \oplus \langle L_3 \rangle, \end{aligned}$$

where the $L_i = m^p(H^i)$, $i = 0, 1, 2, 3$ are Lagrangian in M^0 .

5.1. Algebraic Subvarieties of M

Local descriptions of H^i and L_i are as follows: Let $\{z_1, z_2, z_3\}$ be local holomorphic coordinates on M and $\{w_1, w_2, w_3\}$ be local holomorphic coordinates on M^0 , $w_j = s_j + it_j$, $j = 1, 2, 3$. Then the algebraic subvarieties in M are

$$H^0 = M = \langle z_1, z_2, z_3 \rangle,$$

$$H^1 = \langle z_1, z_2, 0 \rangle,$$

$$H^2 = \langle z_1, 0, 0 \rangle, \text{ and}$$

$$H^3 = \langle 0, 0, 0 \rangle, \text{ which represents a point with } \mathbb{C}\text{-coordinates,}$$

where $H^1 = H$, $H^2 = H \cup H$, $H^3 = H \cup H \cup H$ in the cohomologies. These correspond to complex submanifolds of increasing codimension in M .

5.2. Lagrangian Subvarieties of M^0

The Lagrangian subvarieties in M^0 are

$$L_0 = \langle s_1 - it_1, s_2 - it_2, s_3 - it_3 \rangle,$$

$$L_1 = \langle s_1 - it_1, s_2 - it_2, s_3 + it_3 \rangle,$$

$$L_2 = \langle s_1 - it_1, s_2 + it_2, s_3 + it_3 \rangle, \text{ and}$$

$$L_3 = \langle s_1 + it_1, s_2 + it_2, s_3 + it_3 \rangle,$$

where the generators are taken with real coefficients, so that each L_j is a real three-dimensional submanifold of M^0 . Since $\omega^0 = \frac{i}{2} \sum_{j=1}^n dw_j \wedge d\bar{w}_j$, the restriction $\omega^0|_{L_j} = 0$ and L_j is Lagrangian.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

- [1] Cho, Y.S. (2025) Hodge Classes in Hodge Manifolds. *Advances in Pure Mathematics*, **15**, 518-528. <https://doi.org/10.4236/apm.2025.158026>
- [2] Voisin, C. (2016) The Hodge Conjecture. In: Nash Jr., J.F. and Rassias, M.Th., Eds., *Open Problems in Mathematics*, Springer International Publishing, 521-543. https://doi.org/10.1007/978-3-319-32162-2_17
- [3] Cho, Y.S. (2025) Calabi-Yau Manifolds and Mirror Symmetry. Kyowoo Pub.
- [4] Cho, Y.S. (2021) Complex Manifolds. Kyowoo Pub.
- [5] Cho, Y.S. and Hong, S. (2011) Dynamics of Stringy Congruence in the Early Universe. *Physical Review D*, **83**, Article ID: 104040. <https://doi.org/10.1103/physrevd.83.104040>
- [6] Cox, D. and Katz, S. (1999) Mirror Symmetry and Algebraic Geometry. Vol. 68, American Mathematical Society. <https://doi.org/10.1090/surv/068>
- [7] Weil, A. (1979) Abelian Varieties and the Hodge Ring. In: Weil, A., *Œuvres Scientifiques/Collected Papers. Vol. 3, 1964-1978*, Springer-Verlag, 421-429.
- [8] Weinstein, A. (1971) Symplectic Manifolds and Their Lagrangian Submanifolds. *Advances in Mathematics*, **6**, 329-346. [https://doi.org/10.1016/0001-8708\(71\)90020-x](https://doi.org/10.1016/0001-8708(71)90020-x)
- [9] Ahlfors, L.V. (1978) Complex Analysis. McGraw-Hill.
- [10] Kontsevich, M. (1995) Homological Algebra of Mirror Symmetry. In: Chatterji, S.D., Ed., *Proceedings of the International Congress of Mathematicians*, Birkhäuser, 120-139. https://doi.org/10.1007/978-3-0348-9078-6_11
- [11] Cho, Y.S. (2011) Sheaf Theory. Kyungmoon Pub.
- [12] Greene, B.R. and Plesser, M.R. (1990) Duality in Calabi-Yau Moduli Space. *Nuclear Physics B*, **338**, 15-37. [https://doi.org/10.1016/0550-3213\(90\)90622-k](https://doi.org/10.1016/0550-3213(90)90622-k)