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A Self-Referential Integral Equation for the Nontrivial Zeros of the Riemann Zeta Function

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Abstract

Starting from Jensen's integral representation of the Riemann zeta function, I derive, by a chain of convergent algebraic manipulations, a self-referential integral equation that a complex number F must satisfy in order for $\rho = F(s)/(F(s)-1)$ to be a nontrivial zero of the zeta-function, ζ. The resulting equation, is an explicit kernel and a biconditional characterization of the F-image of the nontrivial zeros. From calculations, I show that the Riemann Hypothesis is equivalent to the statement $|F| = 1$ for every solution. The equation simplifies dramatically to a Dirichlet-like series of lower incomplete gamma functions evaluated on the imaginary axis. A new representation of the Jensen's integral for the Zeta function is show to be reducible to a series relating to Von Staudt-Clausen theorem. These new relations are shown to have a direct bearing on the relationship of the non-trivial zeros of the Zeta function and the primes. I discuss the analytical constraints and outline what is needed to close the gap to a proof of the Riemann Hypothesis.

Keywords

Riemann Hypothesis, Zeta Function, Non-Trivial Zeros, Critical Line, Perron, Fubini Theorem, Primes, Prime Counting, Bernoulli, Entire Functions, Von Staudt-Clausen

1. Introduction

The Riemann Hypothesis (RH) asserts that every nontrivial zero of the Riemann zeta function $\zeta(s)$ lies on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Despite a century of effort, no proof has been found. This paper contributes a new perspective: a rigorous reformulation of the zero condition as a self-referential integral equation in a single complex variable F, from which RH becomes a concrete analytical question about the modulus of solutions.

The key object is the Möbius-type map

$$F(s)=\frac{s}{s-1},$$

which sends the critical line $\sigma=\frac{1}{2}$ to the unit circle $|F|=1$. Under this map, the functional equation of ζ, combined with Schwarz reflection, translates into a closure property on the F-image of the nontrivial zeros.

Starting from Jensen's integral representation of $\zeta(s)$ [1], I systematically derive an integral equation that every zero must satisfy, written purely in F. The derivation is reversible, making it a biconditional: F satisfies the integral equation if and only if $\rho=\frac{F(s)}{F(s)-1}$ is a nontrivial zero of $\zeta(s)$.

2. Jensen's Integral Representation

The starting point was my initial analysis of the Jenson formula in [1] (p 1036). In 2007, I was able to reduce the Jenson Zeta function to a Bernoulli sum. The conversion of the integral form to the Bernoulli form process is included in this paper in SECTION 12. However, due to the seemingly uncontrollable divergence of the sum, I was stalled by the Fubini-interchange theorem and (sum-integral interchangeability, a corollary of Tonelli's Theorem 2.37) in [2]. The hypothesis for the interchange of the Integral and the Sum within the origin work was used to derive Bernoulli representation when the sum was converted to the $\zeta(2n)$ form and subsequently to the Bernoulli($2n$) form. See SECTION 12. The original derived expression was verified by Professor Mark Prevost (LMPA Dunkirk, France). (See the Equation (24) → Equation (25) in this paper). In this paper, I finally overcame this obstacle by recasting the Bernoulli sum as a controlled split between primes and composites. This Bernoulli-form lead to the Prime counting function discussed in SECTION 12.

The Jenson identity is valid for all $s\in\mathbb{C}$ with $s\neq 1$:

$$\boxed{\zeta(s)=\frac{2^{s-1}s}{(2^s-1)(s-1)}+\frac{2^s}{2^s-1}\int_0^\infty\frac{\sin(s\cdot\arctan x)(1+x^2)^{-s/2}}{e^{\pi x}-1}dx.} \tag{1}$$

The integrand is a single closed-form function and there is no infinite series inside the integral. Convergence of the integral is straightforward: near $x=0$, $\sin(s\cdot\arctan x)\sim sx$, and $\frac{1}{(e^{\pi x}-1)}\sim\frac{1}{(\pi x)}$, giving a bounded integrand; near $x=\infty$, exponential decay dominates.

3. Derivation of the Self-Referential Zero Equation

By convergent manipulations only, I derive the self-referential integral equation for $F(s)$ at a nontrivial zero. Each step is reversible to (1) as demonstrated later.

3.1. Step 1: Substitute the Arctan Series

$$\arctan x = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)x^{2k+1}},\ |x|>1 \tag{2}$$

Substituting into (1) gives,

$$s \cdot \arctan x = \frac{s\pi}{2} - s \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)x^{2k+1}},\ \{s \in \mathbb{C}, s \neq 1\} \tag{3}$$

The integrand of (**1**) carries a term $\sin\left(\frac{s\pi}{2} - \Sigma(x,s)\right)$, where $\sum(x,s) = \sum_{k=0}^{\infty} \frac{s(-1)^k}{(2k+1)x^{2k+1}}$.

3.2. Step 2: Convert Sine to Exponentials

Using $\sin(z) = \frac{\left(e^{iz} - e^{-iz}\right)}{(2i)}$:

$$\sin(s \cdot \arctan x) = \frac{1}{2i}\left[e^{\frac{is\pi}{2}} e^{-i\Sigma(x,s)} - e^{-\frac{is\pi}{2}} e^{i\Sigma(x,s)} \right] \tag{4}$$

were, $\Sigma(x,s) = \sum_{k=0}^{\infty} \frac{s(-1)^k}{(2k+1)x^{2k+1}}$.

3.3. Step 3: Absorb All Factors into the Exponents

Using $\left(1+x^2\right)^{-\frac{s}{2}} = \exp\left[-\left(\frac{s}{2}\right)\ln\left(1+x^2\right)\right]$ and $\frac{1}{(2i)} = \left(\frac{1}{2}\right)e^{-i\pi/2}$, absorbs every polynomial/exponential prefactor into exponential form. After these rearrangements,

$$\zeta(s) = \frac{2^{s-1}s}{\left(2^s-1\right)(s-1)} + \frac{2^{s-1}}{2^s-1}\int_0^{\infty} \frac{e^{\tilde{E}_+(x,s)} + e^{\tilde{E}_-(x,s)}}{e^{\pi x}-1}\mathrm{d}x,\ \{s \in \mathbb{C}, s \neq 1\} \tag{5}$$

with

$$\tilde{E}_+(x,s) = \frac{i\pi(s-1)}{2} - i\left(\Sigma(x,s)\right) - \frac{s}{2}\ln\left(1+x^2\right),\ \{s \in \mathbb{C}, s \neq 1\} \tag{6}$$

$$\tilde{E}_-(x,s) = -\frac{i\pi(s+1)}{2} + i\Sigma(x,s) - \frac{s}{2}\ln\left(1+x^2\right),\ \{s \in \mathbb{C}, s \neq 1\} \tag{7}$$

The sign between the exponentials has been absorbed as $e^{-i\pi} = -1$.

3.4. Step 4: Factor *s* from the Exponents

Extract s and rewrite each exponent as a constant (purely imaginary) piece plus s times a kernel function of x. After pulling the remaining constant phases $\mp\frac{i\pi}{2}$ into the s bracket as $\mp\frac{i\pi}{(2s)}$:

$$\tilde{E}_{\pm}(x,s) = -s\,Q_{\pm}(x,s),\ \{s\in\mathbb{C}, s\neq 1\} \tag{8}$$

$$Q_{+}(x,s) = \frac{i\pi}{2s} - \frac{i\pi}{2} + i\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)x^{2k+1}} + \frac{1}{2}\ln\left(1+x^2\right),\ \{s\in\mathbb{C}, s\neq 1\} \tag{9}$$

$$Q_{-}(x,s) = -\frac{i\pi}{2s} + \frac{i\pi}{2} - i\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)x^{2k+1}} + \frac{1}{2}\ln\left(1+x^2\right),\ \{s\in\mathbb{C}, s\neq 1\} \tag{10}$$

3.5. Step 5: Collect the $\frac{i\pi}{2s}$ Structure

Note that $\frac{i\pi}{(2s)} - \frac{i\pi}{2} = -\frac{i\pi(s-1)}{(2s)}$. Introduce $F(s) = \frac{s}{(s-1)}$. Then $\frac{1}{F(s)} = \frac{(s-1)}{s}$ and $-\frac{i\pi(s-1)}{(2s)} = -\frac{i\pi}{(2F(s))}$. Similarly for the opposite sign. Define:

$$P(x,F) = -\frac{i\pi}{2F} + i\left[\frac{\pi}{2} - \arctan x\right] + \frac{1}{2}\ln\left(1+x^2\right), \tag{11}$$

where (2) is used in the reverse direction to write the sum in closed form. Define the "formal conjugate"

$$\overline{P(x,F)} = +\frac{i\pi}{2F} - i\left[\frac{\pi}{2} - \arctan x\right] + \frac{1}{2}\ln\left(1+x^2\right), \tag{12}$$

where all signs of imaginary terms are flipped, but the real ln term is unchanged. For real s, *this equals the true complex conjugate; for complex* s, *it differs. Absolute integrability of both sides against* $\frac{1}{e^{\pi x}-1}$ *on* $(0, \infty)$ *yields* (13) *as an identity of meromorphic functions on* $\{s\in\mathbb{C}, s\neq 1\}$.

Then $Q_{\pm}$ take the compact form $Q_{+}(x,F) = P(x,F)$ and $Q_{-}(x,F) = \overline{P(x,F)}$, and (5) becomes:

$$\zeta(s) = \frac{2^{s-1}}{2^s-1}\left\{F(s) + \int_0^{\infty}\frac{e^{-sP(x,F)} + e^{-s\overline{P(x,F)}}}{e^{\pi x}-1}\mathrm{d}x\right\},\ \{s\in\mathbb{C}, s\neq 1\} \tag{13}$$

3.6. Step 6: Impose $\zeta(\rho) = 0$

At a nontrivial zero ρ, since the prefactor $\frac{2^{\rho-1}}{(2^{\rho}-1)}$ is nonzero, the braces in (13) must vanish:

$$F(\rho) = -\int_0^{\infty}\frac{e^{-\rho P(x,\rho)} + e^{-\rho\overline{P(x,\rho)}}}{e^{\pi x}-1}\mathrm{d}x \tag{14}$$

Note: Perron gives, at zeros, ρ

$$\psi_0\left(\frac{1}{ix}\right) = \frac{1}{ix} - \sum_{\rho}\frac{e^{\frac{i\pi\rho}{2}}x^{-\rho}}{\rho} - \log(2\pi) - \frac{1}{2}\ln\left(1+x^2\right),$$

$$P(x,F)+\frac{i\pi}{2F}-i\left[\frac{\pi}{2}-\arctan x\right]=\frac{1}{2}\ln\left(1+x^2\right),$$

$$P(x,F)=\frac{1}{ix}-\psi_0\left(\frac{1}{ix}\right)-\sum_\rho\frac{\mathrm{e}^{\frac{i\pi\rho}{2}}x^{-\rho}}{\rho}-\log(2\pi)-\frac{i\pi}{2F(s)}+i\left[\frac{\pi}{2}-\arctan x\right],$$

$$P(x,F)=\frac{1}{ix}-\psi_0\left(\frac{1}{ix}\right)-\sum_\rho\frac{\mathrm{e}^{\frac{i\pi\rho}{2}}x^{-\rho}}{\rho}-\log(2\pi)-\frac{i\pi}{2\rho}-i\arctan x,$$

$$P\left(\frac{1}{ix},F\right)=x-\psi_0(x)-\sum_\rho\frac{x^{-\rho}}{\rho}-\log(2\pi)-\frac{i\pi}{2\rho}-i\arctan x,$$

$$\psi_0(1-ix)=(1-ix)-i\sum_\rho\frac{(1-ix)^{-\rho}}{\rho}-\log(2\pi)-\frac{1}{2}\ln\left(1-(1-ix)^{-2}\right)$$

Using the identity $(1-ix)^\rho=i\mathrm{e}^{\rho P(x,F(\rho))}\to\rho\log(1-ix)=\frac{\pi}{2}+\rho P(x,F(\rho))$

$$\to\frac{\pi}{2}=\rho\left(\log(1-ix)-P(x,F(\rho))\right)$$

$$=\rho\left(\log(1-ix)+\frac{i\pi}{2F}-i\left[\frac{\pi}{2}+\arctan x\right]-\frac{1}{2}\ln\left(1+x^2\right)\right)$$

$$\frac{\pi}{2}=\rho\left(\log(1-ix)+\frac{i\pi}{2F}-i\left[\frac{\pi}{2}+\arctan x\right]-\frac{1}{2}\ln\left(1+x^2\right)\right),$$

one gets:

$$\psi_0(1-ix)=(1-ix)-i\sum_\rho\frac{\mathrm{e}^{\rho P(x,\rho)}x^{-\rho}}{\rho}-\log(2\pi)-\frac{1}{2}\ln\left(1-(1-ix)^{-2}\right)$$

Riemann's original strategy from 1859 [3] was to use the explicit formula and analyze the sum over zeros. It's been studied for 165 years. The connection through the $P(x,\rho)$ kernel is genuinely new notation, but the underlying object (sum of $\frac{y^\rho}{\rho}$ over zeros) is classical. The fact that one can rewrite $\frac{y^\rho}{\rho}$ as $(-i)\mathrm{e}^{\rho P(x,\rho)}$ is an algebraic reformulation. It might be useful for finding new angles, but it doesn't change the analytical hardness.

3.7. Step 7: Close the Self-Reference via $\rho=\frac{F(\rho)}{F(\rho)-1}$

From $F(\rho)=\frac{\rho}{(\rho-1)}$, I solve: $\rho=\frac{F(\rho)}{F(\rho)-1}$. Substituting into (14), eliminate ρ and obtain an equation purely in F:

$$\boxed{F(\rho)=-\int_0^\infty\frac{\exp\left(-\frac{F(\rho)}{F(\rho)-1}P(x,F(\rho))\right)+\exp\left(-\frac{F(\rho)}{F(\rho)-1}\overline{P(x,F)}\right)}{\mathrm{e}^{\pi x}-1}\mathrm{d}x}\quad(15)$$

This is the main result.

Theorem 1 (Biconditional). *For* $F\in\mathbb{C}$ *with* $F\neq1$, *Equation* (15) *holds if* *and only if* $\rho := F/(F-1)$ *is a nontrivial zero of* ζ.

Proof sketch: Every step from (1) to (15) is reversible. Assuming (15), one runs the steps backward to recover $\zeta(\rho)\cdot\left[\dfrac{2^{\rho-1}}{2^{\rho}-1}\right]=0$, and since the bracket is nonzero in the nontrivial range, $\zeta(\rho)=0$. The forward direction is the derivation itself.

4. Simplification via $(x \pm i)^{-\rho}$

Using $\arctan\left(\dfrac{1}{x}\right)=\dfrac{\pi}{2}-\arctan x$ (for $x>0$) and

$\log(x+i)=\left(\dfrac{1}{2}\right)\ln\left(1+x^2\right)+i\arctan\left(\dfrac{1}{x}\right)$ (principal branch):

$$\mathrm{e}^{-\rho P(x,F)}=\mathrm{e}^{\frac{i\pi}{(2(F-1))}}(x+i)^{-\rho}=\mathrm{e}^{\frac{i\pi(\rho-1)}{2}}(x+i)^{-\rho}, \tag{16}$$

using $F-1=\dfrac{1}{(\rho-1)}$. Similarly,

$$\mathrm{e}^{-\rho\overline{P(x,F)}}=\mathrm{e}^{-\frac{i\pi(\rho-1)}{2}}(x-i)^{-\rho},\quad \rho\in\mathbb{C} \tag{17}$$

Substituting into (14) for the principal-value/regularized identity:

$$\frac{\rho}{\rho-1}=-\mathrm{e}^{\frac{i\pi(\rho-1)}{2}}I_{+}(\rho)-\mathrm{e}^{-\frac{i\pi(\rho-1)}{2}}I_{-}(\rho),\{\rho\in\mathbb{C},\text{principal value / regularized}\} \tag{18}$$

were

$$I_{\pm}(\rho)=\int_0^{\infty}\frac{(x\pm i)^{-\rho}}{\mathrm{e}^{\pi x}-1}\mathrm{d}x,\quad \rho\in\mathbb{C} \tag{19}$$

The two integrals satisfy

$$I_{-}(\rho)=\overline{I_{+}(\bar{\rho})},\quad \rho\in\mathbb{C} \tag{20}$$

for all $\rho\in\mathbb{C}$, (the remaining integrand is $O(1)$ at the origin and decays exponentially at infinity).

Remark 1 (Individual divergence, combined convergence). The integrands $\dfrac{(x\pm i)^{-\rho}}{\left(\mathrm{e}^{\pi x}-1\right)}$ are individually non-integrable near $x=0$, where the integrand behaves as $\dfrac{i^{\mp\rho}}{(\pi x)}$. However, the specific combination in (18) cancels at $x=0$: writing $\rho=\sigma+iT$ and expanding near $x=0$, $\mathrm{e}^{-\rho P}+\mathrm{e}^{-\rho\bar{P}}=-i+i=0$, and higher-order terms give integrand $=O(1)$. So (18) should be read as a principal-value/regularized identity, or equivalently via (14) or (15), where the numerator is manifestly bounded.

5. Modulus/Phase Decomposition

Writing $\rho=\sigma+iT$:

$$\mathrm{e}^{\pm\frac{i\pi(\rho-1)}{2}}=\mathrm{e}^{\mp\frac{\pi T}{2}}\cdot\mathrm{e}^{\pm\frac{i\pi(\sigma-1)}{2}}. \tag{21}$$

Define the normalized integrals

$$J_+(\sigma,T):=\mathrm{e}^{-\frac{\pi T}{2}}I_+(\rho),\ J_-(\sigma,T):=\mathrm{e}^{\frac{\pi T}{2}}I_-(\rho). \tag{22}$$

These are bounded: the $\mathrm{e}^{\mp\pi T/2}$ factors compensate the growth/decay of $I_\pm$. Equation (18) becomes:

$$\frac{\sigma+iT}{\sigma-1+iT}=-\mathrm{e}^{\frac{i\pi(\sigma-1)}{2}}J_+(\sigma,T)-\mathrm{e}^{-\frac{i\pi(\sigma-1)}{2}}J_-(\sigma,T). \tag{23}$$

Observation 1. At fixed $T=14.1347\cdots$ (imaginary part of the first nontrivial zero), tabulation of $|J_+(\sigma,T)|$ for $\sigma\in[0.3,0.7]$ gives essentially constant values (within 10^{-5}): $|J_+|\approx 20.271$. The σ-dependence of $J_\pm$ is concentrated almost entirely in their phases, not their moduli.

6. The Critical Line as Self-Conjugate Fixed Point

Proposition 1 (Algebraic properties of F). *For* $F(s)=\frac{s}{(s-1)}$, *the following hold for all* s *where defined*:

1) $F(1-s)=\frac{1}{F(s)}$.

2) $F(\bar{s})=\overline{F(s)}$.

3) *Consequently* $F(1-\bar{s})=\frac{F(s)}{|F(s)|^2}$.

4) $|F(s)|=1\Leftrightarrow \mathrm{Re}(s)=\frac{1}{2}$.

Proof. Direct computation. For (4): $|F|^2=\frac{|s|^2}{|s-1|^2}$. On $\sigma=1/2$, $|s|^2=\frac{1}{4}+T^2=|s-1|^2$, so $|F|=1$. Off $\sigma=\frac{1}{2}$, strict inequality.

Corollary 3 (F-image of zeros). *Let* $\mathcal{F}:=\{F(\rho):\rho\text{ nontrivial zero of }\zeta\}$. *Then* $\mathcal{F}$ *is closed under*:

- $F\mapsto\bar{F}$ (*Schwarz reflection applied to zeros*),
- $F\mapsto\frac{1}{F}$ (Functional-equation $\rho\mapsto 1-\rho$),

consequently, $F\mapsto\frac{1}{\bar{F}}$.

Theorem 2 (RH as a modulus condition). *The Riemann Hypothesis is equivalent to* $\mathcal{F}\subset\{|F|=1\}$, *equivalently to the statement that every solution of* (15) *lies on the unit circle*.

7. Laplace-Transform/Incomplete-Gamma Form

This section develops an alternative representation of the zero equation that connects directly to classical special functions.

Expanding $\frac{1}{\left(e^{\pi x}-1\right)}=\sum_{k=1}^{\infty} e^{-\pi k x}$ (valid for $x>0$, uniformly convergent on $[\varepsilon,\infty)$):

$$I_{\pm}(\rho)=\sum_{k=1}^{\infty}\int_{0}^{\infty}(x\pm i)^{-\rho} e^{-\pi k x}\,dx. \tag{24}$$

The inner Laplace integrals are known in closed form:

$$\int_{0}^{\infty}(x+a)^{-\rho} e^{-sx}\,dx=s^{\rho-1} e^{sa}\,\Gamma(1-\rho,sa), \tag{25}$$

where $\Gamma(\alpha,z)=\int_{z}^{\infty} t^{\alpha-1}e^{-t}\,dt$ is the upper incomplete-gamma-function. With $s=\pi k$, $a=\pm i$, I get

$$\int_{0}^{\infty}(x\pm i)^{-\rho} e^{-\pi k x}\,dx=(\pi k)^{\rho-1} e^{\pm i\pi k}\,\Gamma(1-\rho,\pm i\pi k), \tag{26}$$

and $e^{\pm i\pi k}=(-1)^{k}$.

Substituting into (18):

$$\frac{\rho}{\rho-1}=-\sum_{k=1}^{\infty}(-1)^{k}(\pi k)^{\rho-1}\left[e^{\frac{i\pi(\rho-1)}{2}}\Gamma(1-\rho,i\pi k)+e^{-\frac{i\pi(\rho-1)}{2}}\Gamma(1-\rho,-i\pi k)\right]. \tag{27}$$

7.1. Reduction Using the Functional Equation of ζ

Split $\Gamma(1-\rho,z)=\Gamma(1-\rho)-\gamma(1-\rho,z)$, where $\gamma(\alpha,z)=\int_{0}^{z} t^{\alpha-1}e^{-t}\,dt$ is the lower incomplete gamma. Separating:

$$\begin{aligned}\frac{\rho}{\rho-1}=&-\Gamma(1-\rho)\cdot 2\cos\left(\frac{\pi(\rho-1)}{2}\right)\cdot\pi^{\rho-1}\cdot\eta(1-\rho)\\&+\sum_{k=1}^{\infty}(-1)^{k}(\pi k)^{\rho-1}\left[e^{\frac{i\pi(\rho-1)}{2}}\gamma(1-\rho,i\pi k)+e^{-\frac{i\pi(\rho-1)}{2}}\gamma(1-\rho,-i\pi k)\right],\end{aligned} \tag{28}$$

where $\eta(s)=\left(1-2^{1-s}\right)\zeta(s)$ is the Dirichlet eta function and I used $\sum_{k\geq 1}(-1)^{k+1}k^{\rho-1}=\eta(1-\rho)$. At a nontrivial zero ρ, the functional equation of ζ gives $\zeta(1-\rho)=0$, hence $\eta(1-\rho)=0$. The first term in (28) vanishes, leaving:

$$\frac{\rho}{\rho-1}=\sum_{k=1}^{\infty}(-1)^{k}(\pi k)^{\rho-1}\left[e^{\frac{i\pi(\rho-1)}{2}}\gamma(1-\rho,i\pi k)+e^{-\frac{i\pi(\rho-1)}{2}}\gamma(1-\rho,-i\pi k)\right]. \tag{29}$$

Proposition 2 (Zero-detector). Equation (29) holds if and only if $\zeta(\rho)=0$ (for ρ nontrivial).

7.2. Power-Series Form of the Bracket

Using the Taylor series $\gamma(1-\rho,z)=\frac{\sum_{n=0}^{\infty}(-1)^{n} z^{1-\rho+n}}{\left[n!(1-\rho+n)\right]}$ and combining the two $\pm i$ branches, the $(i)^{n}+(-i)^{n}$ terms vanish for odd n and equal $2(-1)^{\frac{n}{2}}$ for even $n=2m$. After simplification:

$$\frac{\rho}{\rho-1}=2\sum_{k=1}^{\infty}(-1)^k\sum_{m=0}^{\infty}\frac{(-1)^m(\pi k)^{2m}}{(2m)!(1-\rho+2m)}. \tag{30}$$

The outer sum converges conditionally (alternating in k); the inner sum converges absolutely for each fixed k. The order of summation cannot be reversed naively.

8. Analytical Constraints

8.1. Convergence Constraints on the Laplace Transforms

Each individual Laplace integral (26) converges absolutely. The constraints are:

- $\mathrm{Re}(\pi k)>0$: satisfied since $k\ge 1$.
- Branch consistency: $(x+i)^{-\rho}$ on $x\in[0,\infty)$ lies on the line $\mathrm{Im}(z)=1$ in the right half-plane avoiding the principal-branch cut.
- No singularity at $x=0$: each individual Laplace integral is well-behaved because $\mathrm{e}^{-\pi kx}$ is bounded at 0.

8.2. Sum-Integral Interchange

When Inwrite $I_\pm(\rho)=\sum_k\int\cdots$ as in (24), this is conditionally convergent in total, not absolutely. The interchange is justified because:

- Each term of the sum is a convergent integral.
- On $x\in[\varepsilon,\infty)$ for any $\varepsilon>0$, the series $\sum_k \mathrm{e}^{-\pi kx}$ is uniformly convergent and bounded by $1/(\mathrm{e}^{\pi\varepsilon}-1)$, allowing interchange on $[\varepsilon,\infty)$.
- The contribution from $(0,\varepsilon)$ to $I_\pm$ is, individually, divergent; but in the combination $-\mathrm{e}^{\frac{i\pi(\rho-1)}{2}}I_+-\mathrm{e}^{-\frac{i\pi(\rho-1)}{2}}I_-$, the $x\to 0$ singularities cancel.

In practice, (29) avoids these delicate issues entirely: each term is finite, and the cancellation structure has been absorbed into the reduction.

8.3. The Formal versus True Complex Conjugate

In the derivation, $\overline{P(x,F)}$ denotes the formal conjugate (sign flip of imaginary kernel terms), not the true complex conjugate of P (which would involve $\bar{F}$). For real s, these coincide. For complex s, they differ, but the final forms (18), (29) are written in terms of the genuine complex integrals $I_\pm(\rho)$, which satisfy the true conjugacy (20).

8.4. Validity of the Arctan Expansion

The series (2) converges for $|x|>1$; for $|x|<1$, the expansion is

$$\arctan x=\sum_{k\ge 0}\frac{(-1)^k x^{2k+1}}{(2k+1)}.$$

The derivation uses (2), but the closed-form $\sin(s\arctan x)(1+x^2)^{-s/2}$ is analytic across $|x|=1$. The final forms (15) and (18) use closed-form expressions, so no domain-of-validity issue remains.

9. Numerical Verification

All computations were performed by Anthropic Claude in mpmath at 25 - 50 decimal precision.

Identity	Test point	Precision
(1)	$s = 2, 3 + i, \rho_1, \rho_2, -2, -4$	40 digits
(13)	first nontrivial zero	40 digits
(15)	$F(\rho_1) \approx 0.9975 - 0.0707i$	28 digits
(18)	first nontrivial zero	28 digits
$\lvert F(\rho)\rvert = 1 \Leftrightarrow \sigma = 1/2$	multiple T, multiple σ	exact
(29) zero detection	at zeros vs. non-zeros	confirmed

10. What Is Needed to Close the Gap to RH?

Theorem 1 gives a rigorous reformulation of the nontrivial-zero condition as a single integral equation in F. Combined with Proposition 2, the Riemann Hypothesis reduces to:

Every solution of (15) *satisfies* $\lvert F\rvert = 1$.

Equivalently, every ρ satisfying (29) has $\mathrm{Re}(\rho) = \dfrac{1}{2}$.

The open question is therefore concrete and analytical, not combinatorial or number-theoretic. Possible approaches:

10.1. Phase Analysis Approach

Numerical evidence (Section 5) shows that $\lvert J_+(\sigma, T)\rvert$ is remarkably σ-insensitive at fixed T. If this can be proven analytically, (23) reduces to a phase-matching condition that may determine σ uniquely.

I now develop the first of the five approaches outlined in Section 10. The goal is to show that Equation (23) forces $\sigma = 1/2$ by exploiting a near-σ-independence of $\lvert J_+(\sigma, T)\rvert$ at fixed T that emerges from the structure of the integrand.

10.1.1. The Divergence of $I_\pm$ and the Need for Regularization

As noted in the remark after Equation (18), the integrals $I_\pm(\rho)$ defined in (19) are individually non-integrable at $x = 0$. The leading behavior near $x = 0$ comes from the expansion

$$(x+i)^{-\rho} = i^{-\rho}\left(1 - \frac{\rho x}{i} + O(x^2)\right), \tag{31}$$

combined with the Laurent expansion

$$\frac{1}{e^{\pi x} - 1} = \frac{1}{\pi x} - \frac{1}{2} + \frac{\pi x}{12} - \frac{\pi^3 x^3}{720} + O(x^5). \tag{32}$$

Multiplying:

$$\frac{(x+i)^{-\rho}}{e^{\pi x}-1}=\frac{i^{-\rho}}{\pi x}+O(1) \text{ as } x\to 0. \tag{33}$$

Similarly:

$$\frac{(x-i)^{-\rho}}{e^{\pi x}-1}=\frac{(-i)^{-\rho}}{\pi x}+O(1) \text{ as } x\to 0. \tag{34}$$

Using the principal-branch conventions $i=e^{i\pi/2}$ and $-i=e^{-i\pi/2}$:

$$i^{-\rho}=e^{-i\pi\rho/2},\ (-i)^{-\rho}=e^{i\pi\rho/2}. \tag{35}$$

The divergent pieces have residues $i^{-\rho}/\pi$ and $(-i)^{-\rho}/\pi$, both with $\left|i^{\mp\rho}\right|=e^{\pi T/2}$ for $\rho=\sigma+iT$.

10.1.2. Cancellation in the Full Zero Equation

In the combination (18), the phase factors $-e^{\pm i\pi(\rho-1)/2}$ multiply $I_{\pm}(\rho)$. The residue of the divergent piece in the combined integrand is:

$$-\frac{1}{\pi}\left[e^{i\pi(\rho-1)/2}\,i^{-\rho}+e^{-i\pi(\rho-1)/2}\,(-i)^{-\rho}\right]. \tag{36}$$

Substituting (35):

$$e^{i\pi(\rho-1)/2}\,e^{-i\pi\rho/2}=e^{-i\pi/2}=-i, \tag{37}$$

$$e^{-i\pi(\rho-1)/2}\,e^{i\pi\rho/2}=e^{i\pi/2}=+i. \tag{38}$$

The combined residue is $-(-i+i)/\pi=0$: the $1/x$ singularities cancel exactly.

10.1.3. Regularized Individual Integrals

To isolate the finite information in $J_{\pm}(\sigma,T)$ individually, I define:

$$J_{+}^{\mathrm{reg}}(\sigma,T):=e^{-\pi T/2}\left\{\int_0^1\left[\frac{(x+i)^{-\rho}}{e^{\pi x}-1}-\frac{i^{-\rho}}{\pi x}\right]dx+\int_1^{\infty}\frac{(x+i)^{-\rho}}{e^{\pi x}-1}dx\right\}, \tag{39}$$

with analogous definition for J_{-}^{reg}.

Proposition 1 (Regularized decomposition). Equation (23) *may be rewritten in terms of the regularized quantities as*

$$\frac{\sigma+iT}{\sigma-1+iT}=-e^{i\pi(\sigma-1)/2}\,J_{+}^{\mathrm{reg}}(\sigma,T)-e^{-i\pi(\sigma-1)/2}\,J_{-}^{\mathrm{reg}}(\sigma,T), \tag{40}$$

because the regularization counter terms cancel by the same phase identities (37)-(38) *that produce the cancellation in* (18).

10.1.4. Numerical Observation on $J_{+}^{\mathrm{reg}}(\sigma,T)$

Computation of $J_{+}^{\mathrm{reg}}(f(\sigma,T),T)$ over a grid of (σ,T) values yield the following (35-digit precision).

The values agree to 4 - 5 digits across $\sigma\in[0.3,0.7]$. The variation decreases as T increases: at $T=50$, below 10^{-6}; at $T=100$, below 10^{-7}.

σ	$\left\|J_+^{\text{reg}}(\sigma,T)\right\|^2$ at $T=14.134725$
0.3	1.12535027
0.4	1.12533109
0.5	1.12534725
0.6	1.12539876
0.7	1.12548559

Lemma 1: Conjecture 2 (Near-σ-independence of regularized modulus). (Near-σ-independence of regularized modulus). *There exist functions* $g_0:\mathbb{R}_+\to\mathbb{R}_+$ *and* $g_1:\mathbb{R}\times\mathbb{R}_+\to\mathbb{R}$ *such that*

$$\left|J_+^{\text{reg}}\left(f(\sigma,T),T\right)\right|^2=g_0(T)+g_1(\sigma,T),\tag{41}$$

where $g_1\left(\frac{1}{2},T\right)=0$ *and* g_1 *decays uniformly to zero as* $T\to\infty$.

10.1.5. Phase-Matching Strategy

Write $\rho=\sigma+iT$. Define the LHS and RHS of (40):

$$L(\sigma,T):=\frac{\sigma+iT}{\sigma-1+iT},\tag{42}$$

$$R(\sigma,T):=-\mathrm{e}^{i\pi(\sigma-1)/2}J_+^{\text{reg}}\left(f(\sigma,T),T\right)-\mathrm{e}^{-i\pi(\sigma-1)/2}J_-^{\text{reg}}\left(f(\sigma,T),T\right).\tag{43}$$

Separating L into modulus and argument:

$$\left|L(\sigma,T)\right|^2=\frac{\sigma^2+T^2}{(\sigma-1)^2+T^2},\tag{44}$$

$$\arg L(\sigma,T)=\arctan\left(\frac{T}{\sigma}\right)-\arctan\left(\frac{T}{\sigma-1}\right).\tag{45}$$

Write $J_\pm^{\text{reg}}=\left|J_\pm^{\text{reg}}\right|\mathrm{e}^{i\varphi_\pm}$.

Lemma 2 (Modulus of R).

$$\left|R(\sigma,T)\right|^2=\left|J_+^{\text{reg}}\right|^2+\left|J_-^{\text{reg}}\right|^2+2\left|J_+^{\text{reg}}\right|\left|J_-^{\text{reg}}\right|\cos\left(\varphi_+-\varphi_-+\pi(\sigma-1)\right).\tag{46}$$

10.1.6. The Constraint from (40)

At a true zero,

$$L(\sigma,T)=R(\sigma,T),\tag{47}$$

In particular $|L|^2=|R|^2$:

$$\frac{\sigma^2+T^2}{(\sigma-1)^2+T^2}=\left|J_+^{\text{reg}}\right|^2+\left|J_-^{\text{reg}}\right|^2+2\left|J_+^{\text{reg}}\right|\left|J_-^{\text{reg}}\right|\cos\left(\varphi_+-\varphi_-+\pi(\sigma-1)\right).\tag{48}$$

Assuming Conjecture 1 in strong form, both $\left|J_+^{\text{reg}}\right|$ and $\left|J_-^{\text{reg}}\right|$ depend only on T:

$$\frac{\sigma^2+T^2}{(\sigma-1)^2+T^2}=G_0(T)+G_1(T)\cos\left(\varphi_+(\sigma,T)-\varphi_-(\sigma,T)+\pi(\sigma-1)\right),\tag{49}$$

were

$$G_0(T) = \left|J_+^{\text{reg}}(T)\right|^2 + \left|J_-^{\text{reg}}(T)\right|^2 \quad \text{and} \quad G_1(T) = 2\left|J_+^{\text{reg}}(T)\right|\left|J_-^{\text{reg}}(T)\right| \tag{50}$$

10.1.7. The Symmetry *σ* ↔ *1* − *σ*

Under $\sigma \mapsto 1-\sigma$:

$$\left|L(1-\sigma,T)\right|^2 = \frac{(1-\sigma)^2+T^2}{\sigma^2+T^2} = \frac{1}{\left|L(\sigma,T)\right|^2}. \tag{51}$$

Assuming compatible transformation of the regularized quantities:

$$\left|R(1-\sigma,T)\right|^2 = \frac{1}{\left|R(\sigma,T)\right|^2}. \tag{52}$$

Proposition 4 (Fixed point of reflection). If both $L(\sigma,T) = R(\sigma,T)$ and $L(1-\sigma,T) = R(1-\sigma,T)$ hold, then $\left|L(\sigma,T)\right|^2 \cdot \left|L(1-\sigma,T)\right|^2 = 1$.

10.1.8. The Key Equation

Taking the difference of (48) at σ and $1-\sigma$:

$$\frac{\sigma^2+T^2}{(\sigma-1)^2+T^2} - \frac{(\sigma-1)^2+T^2}{\sigma^2+T^2} = G_1(T)\left[\cos\Phi_\sigma - \cos\Phi_{1-\sigma}\right], \tag{53}$$

where $\Phi_\sigma := \varphi_+(\sigma,T) - \varphi_-(\sigma,T) + \pi(\sigma-1)$.

Let $a = \sigma^2+T^2$ and $b = (\sigma-1)^2+T^2$. Then $\frac{a}{b} - \frac{b}{a} = \frac{a^2-b^2}{ab}$, and $a^2-b^2 = (a-b)(a+b) = (2\sigma-1)(2\sigma^2-2\sigma+1+2T^2)$:

$$\frac{a^2-b^2}{ab} = \frac{(2\sigma-1)(2\sigma^2-2\sigma+1+2T^2)}{(\sigma^2+T^2)\left((\sigma-1)^2+T^2\right)}. \tag{54}$$

This vanishes identically if and only if $\sigma = \frac{1}{2}$, (since $2\sigma^2-2\sigma+1+2T^2 > 0$ always).

Theorem 3 (Conditional reduction of RH to a phase identity). *Assume Conjecture* 1 *in its strong form. Then the Riemann Hypothesis is equivalent to the statement: for every* $T > 0$ *and every* $\sigma \in (0,1)$ *with* $\sigma \neq \frac{1}{2}$ *that arises as the real part of some non-trivial zero, the phase identity*

$$G_1(T)\left[\cos\Phi_\sigma - \cos\Phi_{1-\sigma}\right] = \frac{(2\sigma-1)(2\sigma^2-2\sigma+1+2T^2)}{(\sigma^2+T^2)\left((\sigma-1)^2+T^2\right)}$$

must hold. RH is the statement that no nontrivial zero has $\sigma \neq \frac{1}{2}$, *i.e.*, (54) has no solution with $\sigma \neq \frac{1}{2}$ among actual zeros.

10.1.9. Remaining Obstructions

Theorem 3 reduces RH to:

Problem A. Prove Conjecture 1 in strong form $\left|J_{+}^{\mathrm{reg}}(\sigma,T)\right|$ depends only on T, exactly.

Problem B (Phase rigidity). Show that (54) has no solution with $\sigma \neq \frac{1}{2}$ among actual zeros.

10.1.10. Numerical Status and Open Technical Issues

Lemma1: is a conjecture in its strong form, is that

$$\left|J_{+}^{\mathrm{reg}}\left(f(\sigma,T),T\right)\right|^2 := \mathrm{e}^{-\pi T/2}\left\{\int_0^1\left[\frac{(x+i)^{-\rho}}{\mathrm{e}^{\pi x}-1}-\frac{i^{-\rho}}{\pi x}\right]\mathrm{d}x+\int_1^{\infty}\frac{(x+i)^{-\rho}}{\mathrm{e}^{\pi x}-1}\mathrm{d}x\right\},$$

depends only on T for all σ. It does not. As shown by further analysis. So, I have removed it and made σ some function $f(\sigma,T)$:

$$\left|J_{+}^{\mathrm{reg}}\left(f(\sigma,T),T\right)\right|^2 := \mathrm{e}^{-\pi T/2}\left\{\int_0^1\left[\frac{(x+i)^{-\rho}}{\mathrm{e}^{\pi x}-1}-\frac{i^{-\rho}}{\pi x}\right]\mathrm{d}x+\int_1^{\infty}\frac{(x+i)^{-\rho}}{e^{\pi x}-1}\mathrm{d}x\right\},$$

The close values of $\left|J_{+}^{\mathrm{reg}}\left(f(\sigma,T),T\right)\right|$ for $f(\sigma,T)\in\{0.3,0.4,0.5,0.6,0.7\}$ calculated for $T=14.134725$ a root, to the right give,

$$\left|J_{+}^{\mathrm{reg}}\left(f(\sigma,T),T\right)\right|^2$$
$$=1.12535027, 1.12533109, 1.12534725, 1.12539876, 1.12548559.1.1253503.$$

Analytical proof of strong $f(\sigma,T)$-independence is required before Theorem 3 applies unconditionally. Equation (52) was asserted on the basis of "compatible transformation". The regularization (39) subtracts a σ-dependent singularity; verifying that $J_{+}^{\mathrm{reg}}(1-\rho)$ relates to $J_{+}^{\mathrm{reg}}(\rho)$ via the expected reflection is a technical step not yet carried out. The residual σ-variation $G_1(T)$ in Conjecture 1 may contain precisely the information needed to force $\sigma=\frac{1}{2}$ through a different mechanism than the strong-version argument. Understanding $G_1(T)$ is a meaningful subproblem. The phase functions $\varphi_{\pm}(\sigma,T)$ are not known in closed form. Even granting Conjecture 1, Problem B is a nontrivial analytical task. The argument depends on regularization-scheme choices; showing the conclusion is scheme-independent requires verification.

10.1.11. Summary

The Phase Analysis approach reduces RH, under Conjecture 1, to the rigidity of a specific phase identity. The conjecture is supported numerically but not proven. Future work priorities:

- Analytical proof of Conjecture 1 via change of variables $x=\tan(\theta/2)$ or Mellin-Barnes contour analysis of $I_{\pm}(\rho)$.
- Characterization of $\varphi_{\pm} f(\sigma,T),T$ via asymptotic expansion for large T, connecting to the Riemann-Siegel theta function.
- Rigorous treatment of regularization covariance under $\rho\leftrightarrow 1-\rho$.

10.2. The Perron/Explicit-Formula Connection

The paper's kernel $P(x,F)$, defined in Equation (11), has a structural resemblance to Perron's classical formula and to Riemann's explicit formula for prime counting. In this section I make the connection explicit, deriving clean algebraic identities linking the exponential form $\mathrm{e}^{-sP(x,F)}$ to Riemann's explicit formula, and using these to obtain new representations of P in terms of the Chebyshev counting function ψ_0 and sums over the nontrivial zeros of ζ.

10.2.1. The Key Algebraic Identities

Exact identities that form the bridge between the kernel and Perron-type sums.

Proposition 1 (Exponential form of P). *For* $x>0$ *real and* $s\in\mathbb{C}$ *with* $F=s/(s-1)$,

$$\mathrm{e}^{-s\,P(x,F)}=-i(1-ix)^{-s} \tag{55}$$

$$\mathrm{e}^{+s\,P(x,F)}=i(1-ix)^{+s} \tag{56}$$

Proof. Using $F=\frac{s}{(s-1)}$, one has $\frac{s}{F}=s-1$. Substituting the paper's definition

$$P(x,F)=-i\pi/(2F)+i\left[\frac{\pi}{2}-\arctan x\right]+\frac{1}{2}\ln\left(1+x^2\right):$$

$$\begin{aligned}-s\,P(x,F)&=\frac{is\pi}{2F}-is\left[\frac{\pi}{2}-\arctan x\right]-\frac{s}{2}\ln\left(1+x^2\right)\\&=\frac{i\pi(s-1)}{2}-\frac{is\pi}{2}+is\cdot\arctan x-\frac{s}{2}\ln\left(1+x^2\right)\\&=-\frac{i\pi}{2}+is\cdot\arctan x-\frac{s}{2}\ln\left(1+x^2\right).\end{aligned} \tag{57}$$

Exponentiating:

$$\mathrm{e}^{-s\cdot P(x,F)}=\mathrm{e}^{-i\pi/2}\cdot\mathrm{e}^{is\cdot\arctan x}\cdot\left(1+x^2\right)^{-\frac{s}{2}}=-i\cdot(1-ix)^{-s},$$

where in the last step 1 used $1-ix=\sqrt{1+x^2}\,\mathrm{e}^{-i\arctan x}$ (principal branch, $x>0$). Identity (55) follows by the same computation with s replaced by $1-s$.

10.2.2. Riemann's Explicit Formula

Riemann's explicit formula for the Chebyshev counting function $\psi(x)=\sum_{p^k\le x}\log p$ reads, in its analytic form $\psi_0(x)=\psi(x)-\frac{1}{2}\Lambda(x)\mathbf{1}_{x\in\mathbb{N}}$:

$$\psi_0(x)=x-\sum_{\rho}\frac{x^{\rho}}{\rho}-\log(2\pi)-\frac{1}{2}\log\left(1-x^{-2}\right), \tag{58}$$

valid for real $x>1$, where ρ runs over the nontrivial zeros of ζ. The sum is conditionally convergent when the zeros are taken in order of increasing $|\mathrm{Im}\,\rho|$, symmetrically (*i.e.*, ρ paired with $\bar{\rho}$).

For complex y with $|y|>1$ and $|\arg y|<\pi$, the function $\psi_0(y)$ is defined by analytic continuation via the Perron-type contour integral

$$\psi_0(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{y^s}{s}\mathrm{d}s \tag{59}$$

for $c > 1$, and Equation (57) extends by analytic continuation (though the sum over zeros requires careful interpretation for complex y).

10.2.3. Riemann's Formula at $y = 1 - ix$

Substituting $y = 1 - ix$ (with $x > 0$ real, so $|y| = \sqrt{1+x^2} > 1$ and $|\arg y| < \pi/2$) into (57):

$$\psi_0(1-ix) = (1-ix) - \sum_\rho \frac{(1-ix)^\rho}{\rho} - \log(2\pi) - 1/2\log\left(1-(1-ix)^{-2}\right). \tag{60}$$

Using Proposition 1, identity (55) with $s = \rho$ gives $(1-ix)^\rho = -i\,\mathrm{e}^{\rho P(x,F(\rho))}$, hence

$$\sum_\rho \frac{(1-ix)^\rho}{\rho} = -i\sum_\rho \frac{\mathrm{e}^{\rho P(x,F(\rho))}}{\rho}, \tag{61}$$

where I stress that $F(\rho) = \frac{\rho}{(\rho-1)}$ varies with ρ. Substituting (60) into (59):

$$\psi_0(1-ix) = (1-ix) + i\sum_\rho \frac{\mathrm{e}^{\rho P(x,F(\rho))}}{\rho} - \log(2\pi) - 1/2\log\left(1-(1-ix)^{-2}\right). \tag{62}$$

This is Riemann's explicit formula expressed through the paper's kernel. Then:

Theorem 4 (Perron-P identity). *For $x > 0$ real,*

$$\psi_0(1-ix) = (1-ix) + i\sum_\rho \frac{\mathrm{e}^{\rho P(x,F(\rho))}}{\rho} - \log(2\pi) - 1/2\log\left(1-(1-ix)^{-2}\right), \tag{63}$$

where the sum runs over the nontrivial zeros ρ of ζ, paired symmetrically with their conjugates, and $F(\rho) = \frac{\rho}{(\rho-1)}$.

10.2.4. Inverting the Formula

Equation (62) can be rearranged to express the sum over zeros in terms of $\psi_0(1-ix)$ and elementary functions:

$$i\sum_\rho \frac{\mathrm{e}^{\rho P(x,F(\rho))}}{\rho} = \psi_0(1-ix) - (1-ix) + \log(2\pi) + \frac{1}{2}\log\left(1-(1-ix)^{-2}\right). \tag{64}$$

10.2.5. A Representation of *P*(*x*, *F*) via Riemann's Formula

Riemann's explicit formula applied at a *different* argument, $y = \frac{1}{(ix)} = -\frac{i}{x}$, to obtain a representation of $P(x,F)$ itself.

For $x > 1$, $\left|\frac{1}{(ix)}\right| = \frac{1}{x} < 1$, so (57) does not apply directly. However, the formal identity (extending ψ_0 by analytic continuation or taking the sum over zeros in a regularized sense) reads:

$$\psi_0\left(\frac{1}{ix}\right)=\frac{1}{ix}-\sum_{\rho}\frac{\left(1/(ix)\right)^{\rho}}{\rho}-\log(2\pi)-1/2\log\left(1+x^2\right) \tag{65}$$

where I used $\left(\frac{1}{(ix)}\right)^{-2}=(ix)^2=-x^2$, so $1-\left(\frac{1}{(ix)}\right)^{-2}=1+x^2$. The appearance of $\frac{1}{2}\log\left(1+x^2\right)$ is the critical observation: this is precisely the real part of $P(x,F)$.

Solving (64) for $\frac{1}{2}\log\left(1+x^2\right)$:

$$\frac{1}{2}\log\left(1+x^2\right)=\frac{1}{ix}-\psi_0\left(\frac{1}{ix}\right)-\sum_{\rho}\frac{\left(\frac{1}{(ix)}\right)^{\rho}}{\rho}-\log(2\pi). \tag{66}$$

Substituting (65) into the paper's definition (11) of $P(x,F)$:

Theorem 5 (Explicit-formula representation of P). *For* $x>0$ *real,*

$$P(x,F)=\frac{1}{ix}-\psi_0\left(\frac{1}{ix}\right)-\sum_{\rho}\frac{\mathrm{e}^{-i\pi\rho/2}\,x^{-\rho}}{\rho}-\log(2\pi)-\frac{i\pi}{2F}+i\left[\frac{\pi}{2}-\arctan x\right], \tag{67}$$

where I used the identity $\left(\frac{1}{(ix)}\right)^{\rho}=\mathrm{e}^{-i\pi\rho/2}\,x^{-\rho}$ (*principal branch*).

Thus $P(x,F)$ itself admits a representation in which a sum over the nontrivial zeros of ζ appears explicitly.

10.2.6. The Involute Substitution $x\to\frac{1}{ix}$

The transformation $x\mapsto\frac{1}{(ix)}$ is involutive: applying it twice returns to x. Under this substitution, Theorem 4 yields an unexpectedly clean result.

Theorem 6 (Closed form under substitution). *For* $x>1$ *real,*

$$P\left(\frac{1}{ix},F\right)=\frac{1}{2}\log\left(1-x^{-2}\right)-\frac{i\pi}{2F}+\frac{i\pi}{2}-\operatorname{arctanh}(1/x). \tag{68}$$

Proof. Apply (66) with x replaced by $u=\frac{1}{(ix)}\to-\frac{i}{x}$. Then $\frac{1}{(iu)}=x$, so the right-hand side contains x and $\psi_0(x)$ in place of $\frac{1}{(ix)}$ and $\psi_0\left(\frac{1}{(ix)}\right)$.

Using Riemann's explicit formula (57) at real $x>1$:

$$x-\psi_0(x)-\sum_{\rho}\frac{x^{\rho}}{\rho}-\log(2\pi)=1/2\log\left(1-x^{-2}\right) \tag{69}$$

Thus, the sum over zeros in (66) after substitution combines with $x-\psi_0(x)-\log(2\pi)$ to give $1/2\log\left(1-x^{-2}\right)$. The remaining term $i\left[\pi/2-\arctan(-i/x)\right]$ is evaluated using $\arctan(-i/x)=-i\operatorname{arctanh}\left(\frac{1}{x}\right)$ (valid for $x>1$), yielding $\frac{i\pi}{2}-\operatorname{arctanh}\left(\frac{1}{x}\right)$. Collecting terms gives the stated formula.

Theorem 4 has a remarkable feature: the sum over zeros of ζ has been *absorbed* into the elementary function $\frac{1}{2}\log\left(1-x^{-2}\right)$. The substitution $x \to \frac{1}{(ix)}$ effectively eliminates the appearance of ζ zero set in P.

10.2.7. Exponential Form under Substitution

Combining (67) with the identity $\mathrm{e}^{-sP(x,F)} = -i\left(1-ix\right)^{-s}$ (Proposition 1) applied at $x' = \frac{1}{(ix)} = -\frac{i}{x}$:

Corollary 5 (Exponential closed form). *For* $x>1$ *real and* $s\in\mathbb{C}$ *with*

$$F = \frac{s}{(s-1)},\quad \mathrm{e}^{-sP\left(1/(ix),F(s)\right)} = -i\left(\frac{x}{x-1}\right)^{s} \tag{70}$$

This is consistent with $\mathrm{e}^{-sP(x',F)} = -i\left(1-ix'\right)^{-s}$ for $x' = -\frac{i}{x}$, where $1-ix' = 1-\frac{1}{x} = \frac{(x-1)}{x}$.

10.2.8. Sum over Zeros in Closed Form

Summing (69) over nontrivial zeros ρ (with $F = F(\rho) = \frac{\rho}{(\rho-1)}$ varying with ρ):

$$\sum_{\rho} \frac{\mathrm{e}^{-\rho P\left(1/(ix),F(\rho)\right)}}{\rho} = -i\sum_{\rho} \frac{\left(\frac{x}{(x-1)}\right)^{\rho}}{\rho} \tag{71}$$

The sum on the right is a classical Perron-type sum at argument $y = \frac{x}{(x-1)}$. For $x>2$, I have $1<y<2$, and Riemann's explicit formula (57) applies directly:

$$\sum_{\rho} \frac{\left(x/(x-1)\right)^{\rho}}{\rho} = \frac{x}{x-1} - \psi_0\left(\frac{x}{x-1}\right) - \log(2\pi) - \frac{1}{2}\log\left(\frac{2x-1}{x^2}\right). \tag{72}$$

Here I used

$$1-\left(\frac{x}{(x-1)}\right)^{-2} = 1-\left(\frac{(x-1)}{x}\right)^{2} = \frac{(2x-1)}{x^2} \tag{73}$$

Theorem 8 (P-sum and prime counting). *For* $x>2$ *real,*

$$\sum_{\rho} \frac{\mathrm{e}^{-\rho P\left(1/(ix),F(\rho)\right)}}{\rho} = -i\left[\frac{x}{x-1} - \psi_0\left(\frac{x}{x-1}\right) - \log(2\pi) - \frac{1}{2}\log\left(\frac{2x-1}{x^2}\right)\right] \tag{74}$$

Equation (74) directly connects a sum over ζ's nontrivial zeros, involving the paper's kernel P evaluated at $\frac{1}{(ix)}$ with zero-dependent F, to the *Chebyshev prime-counting function* ψ_0 evaluated at $\frac{x}{(x-1)}$, plus elementary corrections.

10.2.9. Structural Interpretation

Heorems 4, 5, 6, and 8 exhibit the paper's self-referential kernel P as a natural object in the Riemann-Perron framework. Specifically:

1) The exponential e^{-sP} is algebraically equivalent to $(1-ix)^{-s}$ (up to phase), which is the natural "shifted Perron" kernel on the line $\operatorname{Re}(z)=1$.

2) Riemann's explicit formula applied at $y=1-ix$ yields a sum over zeros that is exactly $-i$ times the sum of $\mathrm{e}^{\rho P(x,F(\rho))}/\rho$.

3) The substitution $x \to \frac{1}{(ix)}$ trades the sum over zeros in P for an elementary logarithm, revealing an *involute duality* under which the zero-set contribution collapses.

4) The same substitution, applied to the exponential form, produces the Perron sum $\frac{\sum_\rho \left(\frac{x}{(x-1)}\right)^\rho}{\rho}$, which is expressible in closed form through the prime-counting function ψ_0.

10.2.10. Connection to the Riemann Hypothesis

Equation (74) makes explicit the link between the paper's kernel and prime counting. The Riemann hypothesis is equivalent to the error bound

$$\psi_0(y)-y=O\left(\sqrt{y}\log^2 y\right) \text{ as } y\to\infty. \tag{75}$$

As $x\to 2^+$, the argument $y=\frac{x}{(x-1)}\to\infty$. By (74), the behavior of $\psi_0(y)-y$ is therefore controlled by the sum $\frac{\sum_\rho \mathrm{e}^{-\rho P\left(\frac{1}{(ix)},F(\rho)\right)}}{\rho}$ as x approaches 2 from above.

More precisely, rearranging (72):

$$\psi_0\left(\frac{x}{x-1}\right)-\frac{x}{x-1}=-\log(2\pi)-\frac{1}{2}\log\left(\frac{2x-1}{x^2}\right)+i\sum_\rho \frac{\mathrm{e}^{-\rho P\left(\frac{1}{(ix)},F(\rho)\right)}}{\rho}. \tag{76}$$

RH is equivalent to a specific bound on the P-sum on the right of (76) as $x\to 2^+$.

10.2.11. Caveats and Open Questions

1) Theorem 4 and Equation (62) involve a sum over zeros with $|1-ix|^\rho$ terms. For $\rho=\sigma+iT$, one has $\left|(1-ix)^\rho\right|=\left(1+x^2\right)^{\sigma/2}\mathrm{e}^{T\arctan x}$, which grows exponentially in $|T|$. The series is conditionally convergent only via the Riemann-von Mangoldt pairing of zeros at finite truncation, with slow convergence at complex arguments.

2) Theorem 5 (Equation (66)) similarly uses a sum that, at $y=\frac{1}{(ix)}$ with

$|y|<1$, requires analytic continuation for direct numerical evaluation; the individual y^{ρ} terms do not decay in $|T|$.

3) Theorem 6 (the closed form under substitution) is rigorous and numerically verified to machine precision at multiple test values of *x* and *F*.

4) Theorem 8 (Equation (74)) rests on Riemann's explicit formula at $y=\frac{x}{x-1}$ for $x>2$, where $1<y<2$ and the classical formulation applies rigorously.

5) The question of whether the P-sum in (76) has a structural bound that forces (74) and thus RH remains open. The reformulation (74) is equivalent to the classical explicit formula, rewritten in the paper's notation; it does not bypass the analytical hardness of estimating sums over zeros.

10.2.12. Summary

This section has established three rigorously proven structural identities:

1) The algebraic identity $\mathrm{e}^{-sP(x,F)}=-i(1-ix)^{-s}$ (Proposition 1) is exact, pointwise.

2) The closed-form identity

$P\left(\frac{1}{(ix)},F\right)=1/2\log\left(1-x^{-2}\right)-\frac{i\pi}{(2F)}+\frac{i\pi}{2}-\operatorname{arctanh}\left(\frac{1}{x}\right)$ (Theorem 4) for $x>1$

is exact, and numerically.

3) The prime-counting representation (Theorem 5) relating $\frac{\sum_{\rho}\mathrm{e}^{-\rho P\left(\frac{1}{(ix)},F(\rho)\right)}}{\rho}$ to $\psi_0\left(\frac{x}{(x-1)}\right)$ for $x>2$ rigorous, follows from classical Riemann explicit formula.

These identities situate the paper's self-referential kernel within the Perron/explicit-formula framework and provide new expressions for P in terms of ζ's zeros and the prime-counting function. They establish that the reformulation developed in the paper is compatible with, and potentially complementary to, the classical approach initiated by Riemann. Whether these identities yield a novel attack on the Riemann hypothesis in particular, through the bound on the P-sum implicit in (75) is an open question for future work.

11. The Arctangent-Prime Connection

The paper's kernel $P(x,F)$ contains an arctangent term: $i\left[\frac{\pi}{2}-\arctan x\right]$. This section traces the remarkable fact that the same arctangent function encodes the zero-specific phases in Riemann's explicit formula for prime counting. This connection rigorous, giving a clean identity (Theorem 9 below) linking $\arg F(\rho)$ to the phase $\arctan(2\gamma)$ that appears in von Mangoldt formula for $\psi_0(x)-x$.

11.1. Arctangents in the Kernel

Recall the paper's definition:

$$P(x,F) = 1/2\ln\left(1+x^2\right) + i\left[\frac{\pi}{2} - \arctan x\right] - \frac{i\pi}{2F}. \tag{77}$$

The imaginary part contains $\arctan x$, an angular coordinate associated with the real variable $x \in (0,\infty)$: geometrically, $\arctan x$ is the angle that the point $(1,x)$ subtends at the origin. A second arctangent appears implicitly at nontrivial zeros. For any $\rho \in \mathbb{C}$ with $\operatorname{Re}(\rho) = \frac{1}{2}$, I may write $\rho = 1/2 + i\gamma$ with $\gamma \in \mathbb{R}$, and

$$\arg\rho = \arctan(2\gamma),\ |\rho|^2 = \frac{1}{4} + \gamma^2. \tag{78}$$

11.2. Riemann's Explicit Formula in Arctangent Form

Assuming RH (so all nontrivial zeros are of the form $\rho = \frac{1}{2} + i\gamma$), Riemann's explicit formula

$$\psi_0(x) = x - \sum_\rho \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log\left(1 - x^{-2}\right) \tag{79}$$

can be rewritten using (77). For a zero $\rho = \frac{1}{2} + i\gamma$ with conjugate partner $\bar{\rho} = \frac{1}{2} - i\gamma$,

$$\frac{x^\rho}{\rho} + \frac{x^{\bar{\rho}}}{\bar{\rho}} = \frac{2\sqrt{x}\cos\left(\gamma\log x - \arctan(2\gamma)\right)}{\sqrt{1/4+\gamma^2}}, \tag{80}$$

as follows by writing $\frac{1}{\rho} = \frac{e^{-i\arctan(2\gamma)}}{\sqrt{\frac{1}{4}+\gamma^2}}$ and $x^\rho = \sqrt{x}\,e^{i\gamma\log x}$. Summing over zeros $\gamma > 0$ gives:

$$\psi_0(x) - x = -2\sqrt{x}\sum_{\gamma>0}\frac{\cos\left(\gamma\log x - \arctan(2\gamma)\right)}{\sqrt{1/4+\gamma^2}} - \log(2\pi) - 1/2\log\left(1-x^{-2}\right). \tag{81}$$

The quantity $\arctan(2\gamma)$ is the *zero-specific phase*: it encodes the argument of ρ and determines, along with $\gamma\log x$, the oscillatory contribution of each zero to the prime-counting function.

11.3. The Link to *F*(*ρ*)

The same phase $\arctan(2\gamma)$ appears in the paper's kernel, through the map $F(\rho) = \frac{\rho}{(\rho-1)}$.

Theorem 9 (Arctangent identity for $\arg F$). *Let $\rho = \frac{1}{2} + i\gamma$ with $\gamma \in \mathbb{R}$ (a point on the critical line). Then*

$$|F(\rho)| = 1 \text{ and } \arg F(\rho) = 2\arctan(2\gamma) - \pi. \tag{82}$$

Proof. Write $F(\rho) = \left(\frac{1}{2} + i\gamma\right)\Big/\left(-\frac{1}{2} + i\gamma\right)$. Using $\arg(a+ib) = \arctan(b/a)$

for $a>0$ and the appropriate branch convention for $a<0$:

$$\arg\left(\frac{1}{2}+i\gamma\right)=\arctan\left(\frac{\gamma}{\left(\frac{1}{2}\right)}\right)=\arctan(2\gamma), \tag{83}$$

$$\arg\left(-\frac{1}{2}+i\gamma\right)=\pi-\arctan(2\gamma)\ \left(\text{second quadrant for }\gamma>0\right). \tag{84}$$

Hence

$$\begin{aligned}\arg F(\rho)&=\arg\left(\frac{1}{2}+i\gamma\right)-\arg\left(-\frac{1}{2}+i\gamma\right)\\&=\arctan(2\gamma)-\left(\pi-\arctan(2\gamma)\right)\\&=2\arctan(2\gamma)-\pi.\end{aligned} \tag{85}$$

For the modulus, $|F(\rho)|^2=\frac{\left(\frac{1}{4}+\gamma^2\right)}{\left(\frac{1}{4}+\gamma^2\right)}=1$, as established in Proposition 1 of the paper.

Corollary 10 (Kernel phase at critical-line zeros). *For* $\rho=\frac{1}{2}+i\gamma$,

$$-\frac{i\pi}{2F(\rho)}=\frac{i\pi}{2}\mathrm{e}^{-2i\arctan(2\gamma)}=\frac{i\pi}{2}\cdot\frac{\left(\frac{1}{2}-i\gamma\right)^2}{\frac{1}{4}+\gamma^2}. \tag{86}$$

Proof. From Theorem 1, $F(\rho)=\mathrm{e}^{i(2\arctan(2\gamma)-\pi)}=-\mathrm{e}^{2i\arctan(2\gamma)}$. Hence $\frac{1}{F(\rho)}=-\mathrm{e}^{-2i\arctan(2\gamma)}$, and

$$-\frac{i\pi}{2F(\rho)}=\frac{i\pi}{2}\mathrm{e}^{-2i\arctan(2\gamma)} \tag{87}$$

For the second form, use $\mathrm{e}^{-i\arctan(2\gamma)}=\left(\frac{1}{2}-i\gamma\right)\Big/\sqrt{\frac{1}{4}+\gamma^2}$, so

$$\mathrm{e}^{-2i\arctan(2\gamma)}=\frac{\left(\frac{1}{2}-i\gamma\right)^2}{\left(\frac{1}{4}+\gamma^2\right)} \tag{88}$$

11.4. The Structural Convergence

Theorem 9 and Equation (87) exhibit a shared arctangent structure. Specifically:

1) In the prime-counting explicit formula (77), the phase attached to each zero $\rho=\frac{1}{2}+i\gamma$ is $-\arctan(2\gamma)$, coming from $\arg\left(\frac{1}{\rho}\right)=-\arg\rho=-\arctan(2\gamma)$.

2) In the paper's kernel $P(x,F(\rho))$ at a critical-line zero, the "F-phase"

$-\frac{i\pi}{(2F(\rho))}$ carries the factor $\mathrm{e}^{-2i\arctan(2\gamma)}$, *twice* the arctangent of (1).

3) The factor of 2 comes from the Möbius identity $F(\rho)=\frac{\rho}{(\rho-1)}$, which squares the phase relative to ρ itself.

Then,

Theorem 11 (Shared arctangent structure). For any critical-line zero $\rho=1/2+i\gamma$:

$$\arg\left(\frac{1}{\rho}\right)=-\arctan(2\gamma)\quad\text{(explicit formula phase)},$$

$$\arg\left(\frac{1}{F(\rho)}\right)=-2\cdot\arctan(2\gamma)+\pi\quad\text{(kernel phase, twice the zero angle)}.$$

In particular,

$$\arg\left(\frac{1}{F(\rho)}\right)=2\cdot\arg\left(\frac{1}{\rho}\right)+\pi\ .$$

11.5. Interpretation: From Individual Zeros to Collective Phases

The explicit formula (80) may be rewritten, using (77), as

$$\psi_0(x)-x=-2\sqrt{x}\sum_{\gamma>0}\frac{\cos(\gamma\log x+\arg(1/\rho))}{|\rho|}+O(1),\tag{89}$$

where $\rho=\frac{1}{2}+i\gamma$ and the $O(1)$ captures the elementary corrections. Each zero contributes an oscillation $\frac{\cos\left(\gamma\log x+\arg\left(\frac{1}{\rho}\right)\right)}{|\rho|}$ —a “wave” with frequency γ and phase $\arg\left(\frac{1}{\rho}\right)$.

Under $\rho\mapsto F(\rho)$ (the paper’s Möbius map), each zero’s phase $\arg\left(\frac{1}{\rho}\right)$ maps to $\arg\left(\frac{1}{F(\rho)}\right)=2\arg\left(\frac{1}{\rho}\right)+\pi$: a doubling plus shift. This doubling is characteristic of Möbius transformations that fix the unit circle.

11.6. A Conjectural Reformulation of RH

The paper’s biconditional (Theorem 2) states: RH $\Leftrightarrow F(\mathcal{F})\subseteq\{|F|=1\}$, where $\mathcal{F}$ is the nontrivial-zero set. Combined with Theorem 9, RH is equivalent to:

$$\arg F(\rho)=2\arctan(2\operatorname{Im}(\rho))-\pi\quad\text{for every nontrivial zero }\rho.$$

Since (85) only requires $|F(\rho)|=1$ (which implies $\operatorname{Re}(\rho)=\frac{1}{2}$, so $\operatorname{Im}(\rho)=\gamma$ with $\rho=1/2+i\gamma$), the condition reads: the only constraint on each zero’s phase is the value of its imaginary part.

11.7. Connection to the Perron-P Identity

Combining Theorem 9 with the Perron-P identity of Section 10.2 (Equation (62)):

$$\psi_0(1-ix)=(1-ix)+i\sum_{\rho}\frac{e^{\rho P(x,F(\rho))}}{\rho}-\log(2\pi)-\frac{1}{2}\log\left(1-(1-ix)^{-2}\right), \tag{90}$$

the arctangent content of each term can be made explicit. On the critical line,

$$P(x,F(\rho))=\frac{1}{2}\ln\left(1+x^2\right)+i\arctan\left(\frac{1}{x}\right)+\frac{i\pi}{2}e^{-2i\arctan(2\gamma)}. \tag{91}$$

The kernel thus carries two arctangent contributions: one from the spatial variable x and one from the zero-specific phase $\arctan(2\gamma)$.

11.8. What This Does and Does Not Establish

Established rigorously.

1) Theorem 9: on the critical line, $\arg F(\rho)=2\cdot\arctan(2\gamma)-\pi$.

2) Corollary 10: the kernel's F-dependence is controlled by $e^{-2i\arctan(2\gamma)}$.

3) Theorem 11: the arctangent arctan(2γ) that appears in Riemann's explicit formula (80) is the same arctangent (up to doubling and constant shift) that appears in the paper's kernel P via F.

4. Equation (90): (**Not established).** The critical-line kernel carries explicit arctangent structure, with the zero-specific phase appearing as $e^{-2i\arctan(2\gamma)}$.

1) Whether the arctangent structure forces zeros to lie on the critical line. Off the critical line, a zero $\rho=\sigma+i\gamma$ with $\sigma\neq\frac{1}{2}$ has $\arg\left(\frac{1}{\rho}\right)=-\arctan\left(\frac{\gamma}{\sigma}\right)$, still a well-defined arctangent. The arctangent structure does not inherently distinguish $\sigma=\frac{1}{2}$.

2) Whether the shared arctangent suggests a new bound on $\psi_0(x)-x$. The identities (80)-(81) are algebraic rewritings; they do not provide analytical leverage on the conditional sum over zeros.

11.9. Open Directions

The shared arctangent structure suggests the following avenues:

1) Fourier-theoretic interpretation. The explicit formula (88) views $\psi_0(x)-x$ as a Fourier-like sum over frequencies γ with phases $\arg\left(\frac{1}{\rho}\right)$. The paper's kernel $P(x,F)$ can be viewed as a "Fourier integrand" with $\arctan\left(\frac{1}{x}\right)$ as the spatial phase and $\arctan(2\gamma)$ as the zero phase. A rigorous Fourier-theoretic identity between these two structures would yield new information.

2) Transformed-domain analysis. Equation (80) suggests changing variables from x to $\log x$ (so $\gamma\log x$ is the Fourier frequency) and from ρ to $\arg\left(\frac{1}{\rho}\right)=-\arctan(2\gamma)$ (so the zero is parametrized by its phase). In this transformed domain, RH reads: for each Fourier frequency γ, the corresponding zero phase is exactly $-\arctan(2\gamma)$.

3) Critical-line characterization via phase. Theorem 9 characterizes the critical line as the locus where $\arg F = 2 \cdot \arctan(2\,\mathrm{Im}(s)) - \pi$. Off the critical line, $\arg F$ takes other values. Can the self-referential integral Equation (15) be shown to force $\arg F(\rho) = 2 \cdot \arctan(2\,\mathrm{Im}\,\rho) - \pi$ at zeros? This would be equivalent to RH.

11.10. Summary

This section has established a precise identity (Theorem 9) linking the phase of $F(\rho)$ at a critical-line zero to the arctangent phase $\arctan(2\gamma)$ that appears in Riemann's explicit formula for prime counting. The identity makes rigorous a structural resemblance between the paper's Möbius-based kernel and the classical zero-phase analysis of $\psi_0(x) - x$. The resemblance is real and, has not been recorded in this form. Whether it opens a new attack on RH remains an open question; the identities in this section are algebraic consequences of the Möbius structure and of well-known phase relations, rather than independent analytical facts about the distribution of zeros.

12. The Prime-Bernoulli Bridge

This section establishes a structural bridge between two apparently unrelated objects: (a) the exact prime-counting formula arising from the Von Staudt-Clausen theorem and Wilson's theorem, and (b) the divergent Bernoulli series obtained in the original derivation of the zero equation (Equation (112) of the paper, which I will call the wall). The arithmetic of Bernoulli numbers modulo 1 provides a canonical splitting of the wall into a convergent sum indexed by primes, and a divergent sum indexed by certain integers.

12.1. Prime Counting via Bernoulli Numbers

Theorem 12 (See [4] Exact prime counting via Bernoulli arctangent). *For p an odd prime, let $\pi(p-1)$ denote the number of primes strictly less than p. Then*

$$\pi(p-1) = \frac{1}{\pi} \sum_{n=1}^{(p-1)/2} \arctan\left(\tan\left(\pi B_{2n} (2n)!\right)\right) \cdot (2n+1), \tag{92}$$

where B_{2n} is the $2n$-th Bernoulli number.

Proof. The map $x \mapsto \arctan(\tan(\pi x))/\pi$ is the fractional-part function centered in $(-1/2, 1/2]$. The Von Staudt-Clausen theorem asserts

$$B_{2n} + \sum_{\substack{q \text{ prime} \\ (q-1)|2n}} \frac{1}{q} \in \mathbb{Z} \tag{93}$$

Multiplying by $(2n)!$:

$$B_{2n}(2n)! = (\text{integer}) - \sum_{\substack{q \text{ prime} \\ (q-1)|2n}} \frac{(2n)!}{q}. \tag{94}$$

For primes $q \le 2n$ satisfying $(q-1)\,|\,2n$, the term $(2n)!/q$ is an integer.

The only possible non-integer contribution comes from $q=2n+1$ (which satisfies $(q-1)|2n$ trivially), and is included if $2n+1$ is prime.

When 2n + 1 is prime, Wilson's theorem gives $(2n)!\equiv -1(\bmod 2n+1)$, so $(2n)!/(2n+1)$ has fractional part $-1/(2n+1)(\bmod 1)$, equivalently $(2n)/(2n+1)$. Subtracting this from an integer, $B_{2n}(2n)!$ has centered fractional part exactly $1/(2n+1)$. When $2n+1$ is composite, $B_{2n}(2n)!$ is an integer.

Therefore

$$\frac{1}{\pi}\arctan\left(\tan\left(\pi B_{2n}(2n)!\right)\right)=\begin{cases}1/(2n+1) & \text{if } 2n+1 \text{ prime,}\\ 0 & \text{otherwise.}\end{cases} \tag{95}$$

Multiplying by $(2n+1)$ and summing from $n=1$ to $(p-1)/2$ counts the primes in $\{3,5,7,\cdots,p\}$, which is $\pi(p-1)$.

Remark. Theorem 12 is a Bernoulli-based reformulation of classical prime-counting formulas via Wilson's theorem. It is of expository rather than computational interest: the required precision in $B_{2n}(2n)!$ is on the order of $\log_{10}(\text{denominator})$, which grows like $2n\log(2n)$; this makes the formula impractical for large p.

12.2. The Wall Series Revisited

In the original derivation of the paper, I reduced the zeta relation (1) to the form of a Bernoulli-type series, by reducing the Jensen formula to a Bernoulli series:

Jensen formula. [1] (p 1036)

$$\zeta(z)=\left(\frac{2^{z-1}}{2^z-1}\right)\left(\frac{z}{z-1}\right)+\frac{2}{2^z-1}\int_0^\infty\left(\frac{\left(\frac{1}{4}+t^2\right)^{-\frac{z}{2}}\sin\left(z\tan^{-1}2t\right)}{e^{2\pi t}-1}\right)\mathrm{d}t \tag{96}$$

Substituting tan θ = 2t,

$$\zeta(z)=\left(\frac{2^{z-1}}{2^z-1}\right)\left(\frac{z}{z-1}\right)+\frac{2}{2^z-1}\int_0^{\frac{\pi}{2}}\left(\frac{\left(\frac{1}{2}\sec z\theta\right)\sin(z\theta)\left(\frac{1}{2}(\sec\theta)^2\right)}{e^{\pi\tan\theta}-1}\right)\mathrm{d}\theta \tag{97}$$

This reduces to

$$\zeta(z)=\left(\frac{2^{z-1}}{2^z-1}\right)\left(\frac{z}{z-1}\right)+\frac{2^z}{2^z-1}\int_0^{\frac{\pi}{2}}\left(\frac{\sin(z\theta)\left((\sec\theta)^{2-z}\right)}{e^{\pi\tan\theta}-1}\right)\mathrm{d}\theta \tag{98}$$

Using the Chebyshev-T function,

$$\cos\left(z\cos^{-1}(x)\right)=\frac{1}{2}\left[\left(x+i\sqrt{1-x^2}\right)^z+\left(x-i\sqrt{1-x^2}\right)^z\right]^2 \tag{99}$$

Putting $x=\cos\theta$, and expand for $\sin(z\theta)$,

$$\zeta(z)=\left(\frac{2^{z-1}}{2^{z}-1}\right)\left[\left(\frac{z}{z-1}\right)+2z\int_{0}^{\frac{\pi}{2}}\left(\frac{(\cos\theta)^{z-2}\left(1-\frac{1}{2}\left(\left(\cos\theta+i\left(1-\cos\theta^{2}\right)^{\frac{1}{2}}\right)\right)^{z}+\frac{1}{2}\left(\left(\cos\theta-i\left(1-\cos\theta^{2}\right)^{\frac{1}{2}}\right)\right)^{z}\right)}{\mathrm{e}^{\frac{\pi\sqrt{1-\cos\theta^{2}}}{\cos\theta}}-1}\right)\mathrm{d}\theta\right] \tag{100}$$

$$\zeta(z)=\left(\frac{2^{z-1}}{2^{z}-1}\right)\left[\left(\frac{z}{z-1}\right)+2s\int_{0}^{\frac{\pi}{2}}\left(\frac{(\cos\theta)^{z-2}\left(1-\frac{1}{2}\left(\left(\cos\theta+i\left(1-\cos\theta^{2}\right)^{\frac{1}{2}}\right)\right)^{z}+\frac{1}{2}\left(\left(\cos\theta-i\left(1-\cos\theta^{2}\right)^{\frac{1}{2}}\right)\right)^{z}\right)}{\mathrm{e}^{\frac{\pi\sqrt{1-\cos\theta^{2}}}{\cos\theta}}-1}\right)\mathrm{d}\theta\right] \tag{101}$$

Which can be reduced again by the substitution, $x=\tan\theta$,

$$\zeta(z)=\left(\frac{2^{z-1}}{2^{z}-1}\right)\left[\left(\frac{z}{z-1}\right)+2z\int_{0}^{\infty}\left(\frac{\left(\frac{1}{\sqrt{1+x^{2}}}\right)^{2z}\left(\left(\frac{1}{\sqrt{1+x^{2}}}\right)^{-2z}-\left(\frac{1}{2}(1+ix)^{z}+\frac{1}{2}(1-ix)^{z}\right)^{2}\right)^{\frac{1}{2}}}{\mathrm{e}^{\pi x}-1}\right)\mathrm{d}x\right] \tag{102}$$

And finally, one arrives at the Abel Plena form [1]:

$$\zeta(s)=\left(\frac{2^{s-1}}{2^{s}-1}\right)\left[\left(\frac{s}{s-1}\right)+2\int_{0}^{\infty}\left(\frac{i(1+ix)^{-s}+(1+ix)^{-s}}{\mathrm{e}^{\pi x}-1}\right)\mathrm{d}x\right] \tag{103}$$

The function (103) can now be reduced to a series form as follows:

$$(1+ix)^{-s}=\sum_{n=0}^{\infty}\frac{(ix)^{n}\Gamma(1-s)}{\Gamma(-s-n+1)n!},\quad (1-ix)^{-s}=\sum_{n=0}^{\infty}\frac{(-ix)^{n}\Gamma(1-s)}{\Gamma(-s-n+1)n!} \tag{104}$$

It is convenient at this point to note that the odd terms vanish and one is left with:

$$\zeta(s)=\left(\frac{2^{s-1}}{2^{s}-1}\right)\left[\left(\frac{s}{s-1}\right)+\int_{0}^{\infty}\left(2i\sum_{n=1}^{\infty}\frac{(ix)^{2n-1}\Gamma(1-s)}{2\Gamma(-s-2n+2)(2n-1)!\mathrm{e}^{\pi x}-1}\right)\mathrm{d}x\right] \tag{105}$$

$$\zeta(s)=\left(\frac{2^{s-1}}{2^{s}-1}\right)\left[\left(\frac{s}{s-1}\right)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\Gamma(1-s)}{\Gamma(-s-2n+2)(2n-1)!}\int_{0}^{\infty}\left(\frac{x^{2n-1}}{\mathrm{e}^{\pi x}-1}\right)\mathrm{d}x\right] \tag{106}$$

Using the relation [1] (p 1038)

$$\zeta(2n)\Gamma(2n)\left[\frac{1}{\pi^{2n}}\right]=\int_{0}^{\infty}\frac{x^{2n-1}}{\mathrm{e}^{\pi x}-1}\mathrm{d}x \tag{107}$$

$$\zeta(s)=\left(\frac{2^{s-1}}{2^{s}-1}\right)\left[\left(\frac{s}{s-1}\right)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2\Gamma(1-s)\zeta(2n)\Gamma(2n)}{\pi^{2n}\Gamma(-s-n+2)(2n-1)!}\mathrm{d}x\right] \tag{108}$$

$$\zeta(s)=\left(\frac{2^{s-1}}{2^{s}-1}\right)\left[\left(\frac{s}{s-1}\right)+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2\Gamma(1-s)\zeta(2n)}{\pi^{2n}\Gamma(-s-n+2)}\mathrm{d}x\right] \tag{109}$$

Using the relation,

$$\zeta(2n)=\frac{(-1)^{n+1}2^{2n-1}\pi^{2n}B_{2n}}{(2n)!} \tag{110}$$

One arrives at the desired form:

$$\zeta(z)=\left(\frac{2^{z-1}}{2^{z}-1}\right)\left[\left(\frac{z}{z-1}\right)+\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}\Gamma(1-z)}{\Gamma(2-z-2n)(2n)!}\right] \tag{111}$$

When the Zeta function vanishes at a root ρ,

$$\frac{\rho}{\rho-1}=-\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}\Gamma(1-\rho)}{\Gamma(2-\rho-2n)(2n)!} \tag{112}$$

This series diverges factorially (**the "wall"** of Approach I first used [4] and [5]). as the Bernoulli numbers grow like $B_{2n}\sim(-1)^{n+1}2(2n)!/(2\pi)^{2n}$, making the terms grow unboundedly once $n\gtrsim e\pi^{2}\approx 23$. Using the Gamma-function identity $\frac{\Gamma(1-\rho)}{\Gamma(-\rho-2n+2)}=-(\rho)_{2n-1}$ (Pochhammer rising factorial), I rewrite (112) as:

$$\frac{\rho}{\rho-1}=-\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}(\rho)_{2n-1}}{(2n)!}, \tag{113}$$

and equivalently (via $B_{2n}=(-1)^{n+1}\cdot 2\zeta(2n)(2n)!/(2\pi)^{2n}$):

$$\frac{\rho}{\rho-1}=\sum_{n=1}^{\infty}\frac{(-1)^{n}\cdot 2\zeta(2n)(\rho)_{2n-1}}{\pi^{2n}}, \tag{114}$$

all interpreted as asymptotic equalities.

12.3. The Splitting via Von Staudt-Clausen

Now, applying the structural fact from the proof of Theorem 12: each $B_{2n}(2n)!$ decomposes as an integer plus a prime-detector.

Lemma 13 (Bernoulli decomposition). For each $n\geq 1$,

$$B_{2n}(2n)!=N_{n}+\frac{\chi_{P}(2n+1)}{2n+1}, \tag{115}$$

where $N_{n}\in\mathbb{Z}$ *and* $\chi_{P}(m)=1$ if m is prime and 0 otherwise.

Dividing (115) by $(2n)!$

$$B_{2n}=\frac{N_{n}}{(2n)!}+\frac{\chi_{P}(2n+1)}{(2n+1)!}. \tag{116}$$

Substituting (116) into the **wall series** (112) gives a formal splitting into two subseries:

$$\frac{\rho}{\rho-1}=T_{\text{int}}(\rho)+T_{\text{prime}}(\rho), \tag{117}$$

were

$$T_{\text{int}}(\rho)=\sum_{n=1}^{\infty}\frac{2^{2n}N_n\Gamma(1-\rho)}{\Gamma(-\rho-2n+2)\left((2n)!\right)^2},\tag{118}$$

$$T_{\text{prime}}(\rho)=\sum_{n=1}^{\infty}\frac{2^{2n}\chi_P(2n+1)\Gamma(1-\rho)}{\Gamma(-\rho-2n+2)(2n)!(2n+1)!}\tag{119}$$

These series converge absolutely for all $s\in C\setminus Z\le 0$ and defines an entire function of s (the poles of $\Gamma(1-s)$ being cancelled by zeros of $\frac{1}{\Gamma(2-s-2n)}$).

12.4. Convergence of the Prime Part

Numerical values of $T_{\text{prime}}(\rho)$.

At the first nontrivial zero $\rho_1=\frac{1}{2}+14.134725\cdots i$:

$$T_{\text{prime}}(\rho_1)\approx -3.6932670416+3.6491483483i,$$

computed to 20+ digits with absolute convergence verified through $p=97$, where the term is of order 10^{-118}. $T_{\text{prime}}(\rho)$ takes specific values at other notable points:

$T_{\text{prime}}(0)=0$,

$T_{\text{prime}}(1)\approx -0.3687857$,

$T_{\text{prime}}(2)=-\sum_{\{p\text{ odd prime}\}}\frac{2^{p-1}}{p!}\approx -0.8127247$,

$T_{\text{prime}}\left(\frac{1}{2}\right)\approx -0.1776047$.

$T_{\text{prime}}(0)=0$ because $(0)_{p-2}$ vanishes for every $p\ge 3$.

12.4.1. Interpretation

The splitting (118) expresses the wall series as a sum of two structurally distinct pieces:

1) $T_{\text{int}}(\rho)$: carries the *integer part* N_n of each $B_{2n}(2n)!$. This subseries is divergent, reflecting the wall's original behavior.

2) $T_{\text{prime}}(\rho)$: carries the *prime-detector* term, indexed only over n for which $2n+1$ is prime. This subseries is absolutely convergent and defines an entire function of ρ.

The arithmetic of Bernoulli numbers modulo 1, specifically the Von Staudt-Clausen + Wilson mechanism that makes Theorem 12 work and acts as a natural regularization of the wall: it isolates a convergent prime-indexed subseries from a *divergent* integer-indexed one.

It is indeed important to note that the series:

$$\frac{\rho}{\rho-1}=\sum_{n=1}^{\infty}\frac{2^{2n}B_{2n}\Gamma(1-\rho)}{\Gamma(-\rho-2n+2)(2n)!}.\tag{120}$$

Can also be represented by the change of variable:

$$\frac{\rho}{\rho-1}=\sum_{n=0}^{1}\frac{2^{2n}B_{2n}\Gamma(1-\rho)}{\Gamma(-\rho-2n+2)(2n)!}. \tag{121}$$

This again becomes non-divergent, and has a simple form. Splitting (121) again:

$$\frac{\rho}{\rho-1}=T_{\text{int}}(\rho)+T_{\text{prime}}(\rho), \tag{122}$$

were

$$T_{\text{int}}(\rho)=\sum_{n=0}^{1}\frac{2^{2n}N_n\Gamma(1-\rho)}{\Gamma(-\rho-2n+2)\left((2n)!\right)^2} \tag{123}$$

$$T_{\text{prime}}(\rho)=\sum_{n=0}^{1}\frac{2^{2n}\chi_P(2n+1)\Gamma(1-\rho)}{\Gamma(-\rho-2n+2)(2n)!(2n+1)!} \tag{124}$$

Lemma 14 (Lemma 13 extended). (Integer-prime decomposition of the wall series.)

For each integer $n\geq 0$, *define the integer-part coefficient* $N_n\in\mathbb{Z}$ *and the prime indicator* $\chi_P(2n+1)\in\{0,1\}$ *by:*

$$N_0=1, \chi_P(1)=0,$$

and for $n\geq 1$, *by the Von Staudt-Clausen decomposition*

$$B_{2n}(2n)!=N_n+\frac{\chi_P(2n+1)}{2n+1}, \tag{125}$$

were $\chi_P(m)=1$, *if m is a prime, and* 0 *otherwise, and* $N_n\in\mathbb{Z}$ *is the integer remainder.*

Define the two component functions:

$$T_{\text{int}}(\rho)=\frac{\rho}{\rho-1}+\sum_{n=1}^{\infty}\frac{2^{2n}N_n\Gamma(1-\rho)}{\Gamma(-\rho-2n+2)\left((2n)!\right)^2} \tag{126}$$

$$T_{\text{prime}}(\rho)=\sum_{n=1}^{\infty}\frac{2^{2n}\chi_P(2n+1)\Gamma(1-\rho)}{\Gamma(-\rho-2n+2)(2n)!(2n+1)!} \tag{127}$$

Then, the series defining $T_{\text{prime}}(\rho)$ *converges absolutely for all* $\rho\in C\setminus\mathbb{Z}\leq 0$ *and defines an entire function of* ρ. *The series defining* $T_{\text{int}}(\rho)$ *diverges factorially as an asymptotic series, regularized through its identification with* $T_{\text{prime}}(\rho)$ *and at every nontrivial zero* ρ *of the Zeta function,* $\zeta(s)$, *the homogeneous identity*

$$\boxed{T_{\text{int}}(\rho)+T_{\text{prime}}(\rho)=0} \tag{128}$$

Remark 1: (Arithmetic origin of $N_n=0$). By the Von Staudt-Clausen theorem, the denominator of $B_2=1/6$ is the product of primes q with $(q-1)|2$, namely $q\in\{2,3\}$, giving denom $B_2=6$. Multiplication by $2!=2$ cancels the factor of 2, leaving $B_2 2!=\frac{1}{3}$ with denominator equal to the prime $2n+1=3$. Hence the entire content of the $n=1$ term in (127) is the prime-detector fraction $\chi_P(3)=\frac{1}{2}$, and $N_1=0$. For $n\geq 2$, $(2n)!$ contains prime factors beyond the denominator of B_{2n}, and $|N_n|$ grows factorially.

Remark 2: The identity $T_{\text{int}}(\rho)+T_{\text{prime}}(\rho)=0$ is rigorous as a statement about the convergent function $T_{\text{prime}}(\rho)$: it assigns a canonical regularized value to the divergent series $T_{\text{int}}(\rho)$ at each nontrivial zero, namely $T_{\text{prime}}(\rho)$. Independent characterization of $T_{\text{int}}(\rho)$ by some Borel summation, a contour integral representation, or an identification with a known special function is required to convert this identity from a definition into a constraint on the location of ρ.

The convergent prime-indexed series $T_{\text{prime}}(\rho)$ is at every nontrivial zero ρ exactly the negative of the regularized integer-indexed part $T_{\text{int}}(\rho)$. The integer-indexed part is the asymptotic value of a series whose coefficients N_n are integers arising from the integer parts of $B_{2n}(2n)!$. In other words, at every zero, the fractional/prime arithmetic of Bernoulli numbers exactly balances the integer arithmetic of Bernoulli numbers. Two different aspects of $B_{2n}(2n)!$ (its fractional part (which encodes primes) and its integer part), are forced into a specific complex-valued balance at each zero. So, every nontrivial zero of $\zeta(s)$ is a point where a specific convergent sum over primes equals a specific (regularized) sum over Bernoulli integer parts.

For each $n\geq 1$, write the Von Staudt-Clausen decomposition of $B_{2n}(2n)!$ as

$$B_{2n}=M_n-\sum_{\substack{p\text{ prime}\\(p-1)|2n}}\frac{1}{p},\quad M_n\in\mathbb{Z},\tag{129}$$

multiply by $(2n)!$,

$$B_{2n}(2n)!=M_n(2n)!-\sum_{\substack{p\text{ prime}\\(p-1)|2n}}\frac{(2n)!}{p}+\frac{\chi_P(2n+1)}{(2n+1)}\tag{130}$$

The integer part is $N_n=M_n(2n)!$, the fractional part is the prime indicator at $(2n+1)$ alone.

$$\boxed{\text{The zero condition links odd primes to zero locations}}\tag{131}$$

Define the sum:

$$S(\rho)=\sum_{p\text{ odd prime}}\frac{2^{p-1}(\rho)_{p-2}}{(p-1)!\,p!}\tag{132}$$

Every non-trivial zero, ρ, of the Zeta function satisfies:

$$T_{\text{int}}(\rho)=S(\rho)_k=\sum_{p\text{ odd prime}}\frac{2^{p-1}(\rho)_{p-2}}{(p-1)!\,p!}\tag{133}$$

where $(\rho)_k=\rho(\rho+1)\cdots(\rho+k-1)$ is the rising Pochhammer symbol. The right-hand side is an explicit convergent series indexed by the odd primes. The left-hand side is an object built from Bernoulli integer-arithmetic and the rational function $\frac{\rho}{\rho-1}$. The equation forces the two to coincide at exactly the zero set of ζ.

The following TABLES show the relationship of the sum to the zeroes of the Zeta function.

For example, for $\rho_k = \frac{1}{2} + i\gamma_k$:

Table 1. Shows the values of $(\rho)_k$ at the first 10 nontrivial zeros of ζ.

k	γ_k	$\operatorname{Re} S(\rho)_k$	$\operatorname{Im} S(\rho)_k$	$\lvert S(\rho)_k \rvert$
1	14.1347251417	+3.6932670416	−3.6491483483	5.191965
2	21.0220396388	+37.9949667534	+17.6303128263	41.886101
3	25.0108575801	+98.8513216831	+81.7313765641	128.263797
4	30.4248761259	+304.3132227457	+345.1648080459	460.157888
5	32.9350615877	+486.7924752219	+606.0739744659	777.362577
6	37.5861781588	+1070.8293201482	+1553.9693074227	1887.192635
7	40.9187190121	+1758.2393694649	+2867.2657610298	3363.423646
8	43.3270732809	+2418.2384094418	+4342.6536965249	4970.565172
9	48.0051508812	+4005.9619165493	+9144.8376825371	9983.776195
10	49.7738324777	+4610.4350184051	+11869.1449163667	12733.134418

Each row gives the value of S at the k^{th} *nontrivial zero* $\rho_k = \frac{1}{2} + i\gamma_k$.

Table 2. Shows the prime-by-prime contributions to $S(\rho_1)$.

p (odd-prime)	**Re(term)**	**Im(term)**	**|term|**	**cumulative |error|**
3	+0.16666667	+4.71157505	4.71452e+00	9.07406e+00
5	−4.98434470	−15.23726942	1.60318e+01	1.09418e+01
7	+8.38196187	+7.11319129	1.09934e+01	2.69514e−01
11	+0.11766387	−0.24791268	2.74418e−01	1.59713e−02
13	+0.01132720	+0.01127899	1.59850e−02	1.39819e−05
17	−0.00000803	−0.00001141	1.39500e−05	2.42983e−07
19	+0.00000018	−0.00000016	2.43010e−07	3.27315e−11
23	−3.27e−11	+3.27e−12	3.27315e−11	1.05160e−17
29	+1.05e−17	−1.05e−18	1.04856e−17	5.09905e−20
31	+5.10e−20	−5.10e−21	5.09905e−20	2.53577e−27
37	−2.54e−27	+2.54e−28	2.53577e−27	1.87423e−32
41	−1.87e−32	−1.87e−33	1.87086e−32	4.33038e−35
43	−4.33e−35	−4.33e−36	4.33037e−35	1.73301e−40
47	+1.73e−40	−1.73e−41	1.73301e−40	7.16413e−49

Term for prime p: $\frac{2^{p-1}(\rho)_{\{p-2\}}}{[(p-1)!\cdot p!]}$ *evaluated at* $\rho_1 = \frac{1}{2} + 14.134725\cdots i$; Final value: $S(\rho_1) = 3.6932670416 - 3.6491483483i$ (convergence to $\sim 10^{-49}$ by $p \le 47$).

Table 3. Shows the convergence of $S(\rho_1)$, partial sums by primes used.

Primes used	Re partial S	Im partial S	|error vs final|
{3}	+0.16666667	+4.71157505	9.07406e+00
{3, 5}	−4.81767804	−10.52569437	1.09418e+01
{3, 5, 7}	+3.56428383	−3.41250309	2.69514e-01
{3, 5, 7, 11}	+3.68194770	−3.66041577	1.59713e-02
{3, 5, 7, 11, 13}	+3.69327489	−3.64913678	1.39819e-05
{3, 5, 7, 11, 13, 17}	+3.69326686	−3.64914819	2.42983e-07
{3, 5, 7, 11, 13, 17, 19}	+3.69326704	−3.64914835	3.27315e-11
{3, 5, 7, 11, 13, 17, 19, 23}	+3.69326704	−3.64914835	1.05160e-17
{3, …, 29}	+3.69326704	−3.64914835	5.09905e-20

Truncated $S(\rho_1)$ *computed using only the indicated odd primes; The first five odd primes already give* $S(\rho_1)$ *to* 5 *decimal places. Beyond* $p = 23$, *contributions are below* 10^{-17}.

Table 4. Shows $S(\rho)$ at zeros vs. non-zero test points.

σ	T	Re $S(\rho_k)$	Im $S(\rho_k)$	$\lvert S(\rho_1)\rvert$	Note
0.500	**14.134725**	**+3.6932670**	**−3.6491483**	**5.19196**	ρ_1 **(ZERO)**
0.500	14.500000	+4.4022057	−3.6797767	5.73761	off zero
0.500	15.000000	+5.5016939	−3.6192994	6.58544	off zero
0.500	17.000000	+11.7150920	−1.7096260	11.83918	between zeros
0.500	**21.022040**	**+37.9949668**	**+17.6303128**	**41.88610**	ρ_2 **(ZERO)**
0.300	14.134725	+3.7147204	−3.2897098	4.96199	off line, $T = \gamma_1$
0.700	14.134725	+3.6498747	−4.0246559	5.43318	off line, $T = \gamma_1$
0.500	10.000000	−0.3355801	−1.1518469	1.19974	below first zero
0.500	20.000000	+28.9514692	+9.8383806	30.57746	between zeros

$S(\rho)$ is well-defined at every point in $\mathbb{C}$. Only at zeros of ζ does it equal the regularized integer companion; Highlighted rows are nontrivial zeros of ζ. The value of S varies smoothly off the zero set.

Table 5. Shows the non-prime odd perturbations at ρ_1.

Phantom p	Re (phantom term)	Im (phantom term)	|phantom term|
9	−2.56100881	+0.25284682	2.573460e+00
15	−0.00046688	+0.00033600	5.752117e−04
21	+2.6e−09	+1.8e−09	3.189771e−09
25	−2.7e−13	+2.7e−14	2.702669e−13
27	−1.8e−15	−1.8e−16	1.837719e−15
33	+2.1e−22	+2.1e−23	2.141018e−22
35	+7.8e−25	+7.8e−26	7.848521e−25

Phantom terms that would appear in $S(\rho_1)$ if the indicated non-primes were treated as primes contributing; Their absence is the prime distribution indications; The phantom at $p = 9$ has magnitude 2.5, comparable in size to the contributions of actual small primes. Its absence (because $9 = 3^2$ is not prime) is part of why $S(\rho_1)$ takes the specific value it does.

Table 6. Asymptotic growth of $|S(\rho_k)|$ with zero index k.

k	γ_k	$\lvert S(\rho_k)\rvert$	$\frac{\lvert S\rvert}{\gamma_k}$	$\frac{\lvert S\rvert}{\gamma_k^2}$
1	14.134725142	5.191965	0.367320	0.025987
2	21.022039639	41.886101	1.992485	0.094781
3	25.010857580	128.263797	5.128325	0.205044
4	30.424876126	460.157888	15.124396	0.497106
5	32.935061588	777.362577	23.602888	0.716649
6	37.586178159	1887.192635	50.209751	1.335857
7	40.918719012	3363.423646	82.197677	2.008804
8	43.327073281	4970.565172	114.721923	2.647812
9	48.005150881	9983.776195	207.973020	4.332306
10	49.773832478	12733.134418	255.819851	5.139645

Growth ratios suggest $|S(\rho_k)|$ scales faster than linearly in γ_k but slower than γ_k^2; All values computed at 60-digit internal precision (mpmath); displayed at 6 - 10 decimals.

The tables together demonstrate a concrete numerical relationship between the odd primes and the nontrivial zeros of the Riemann zeta function, through a single convergent series. They demonstrate a concrete numerical relationship between the odd primes and the nontrivial zeros of the Riemann zeta function, through a single convergent series (133).

SUMMARY OF THE TABLES:

Table 1 establishes the basic phenomenon: $S(\rho)$ is well-defined and computable at the first ten nontrivial zeros, and produces a distinct complex number at each. The values are not random; they grow systematically with γ_k (the imaginary part of the zero), from $|S(\rho_1)| \approx 5.19$ up to $|S(\rho_1)| \approx 12733$. Every nontrivial zero stamps the prime-indexed series with its own specific signature.

Table 2 opens the relation at the first zero. The contribution from each individual odd prime to $|S(\rho_1)|$ is shown, and the structure is striking: the first three odd primes (3, 5, 7) carry essentially all the weight, contributing terms of magnitude 4.7, 16.0, and 11.0 respectively. By prime 17, contributions drop below 10^{-5}; by prime 29, below 10^{-17}. The series converges with astonishing speed, not because the primes thin out, but because the factorial denominators $(p-1)!p!$ dominate the polynomial growth of the Pochhammer numerator. This is the analytical mechanism that makes the prime side of the prime-zero balance a rigorously convergent (entire) function.

Table 3 quantifies this convergence: using just $\{3,5,7\}$ already locates $S(\rho_1)$ to within 0.27 of its true value; adding $\{11,13\}$ gets to 5 decimal places; the rest is increasingly fine corrections. The "prime distribution" is relevant to a given zero is overwhelmingly concentrated in a handful of small primes, and that handful expands slowly as one moves to higher zeros (where the convergence saddle point shifts upward).

Table 4 is the conceptual verification. $S(\rho)$ is computed at the first two nontrivial zeros (highlighted bold) and at seven non-zero test points that are points off the critical line, points between zeros, and points below the first zero. The sum is well-defined everywhere; the values vary smoothly. The "zero" rows are not arithmetically distinguished from their neighbors by anything visible in $S(\rho)$ alone. What is special about a zero is not the value of $S(\rho)$ but the equation

$$T_{\text{int}}(\rho) = S(\rho)_k = \sum_{p \text{ odd prime}} \frac{2^{p-1}(\rho)_{p-2}}{(p-1)!\,p!} \tag{134}$$

The zeros are precisely the points where the prime-indexed series equals the integer-indexed companion. The table makes this distinction concrete by showing $S(\rho)_k$ in both regimes.

Table 5 illustrates what "prime distribution shapes the zeros" actually means. The "phantom" terms (non-primes) show what would appear in $S(\rho)_1$ if non-primes (9, 15, 21, …) were prime. The phantom at $p = 9$ has magnitude 2.57, comparable in size to the actual contributions from primes 5 and 7. Its absence from $S(\rho)_1$ is not a small effect; it is one of the structural reasons $S(\rho)_1$ takes the specific value $3.693 - 3.649i$ rather than something else. If 9 were prime, the Equation (134) would have its solutions in different places: the locations of the zeros are sensitive to which integers are prime in this concrete, quantifiable way.

Table 6 documents the asymptotic behavior. The growth $\frac{|S|}{\gamma_k^2}$ rises from 0.37 (at ρ_1) to 256 (at ρ_1), while $\frac{|S(\rho)_k|}{\gamma_k^2}$ rises from 0.026 to 5.14. So $|S(\rho)_k|$ grows faster than linearly in γ_k but slower than quadratically characteristic γ_k^2 near each zero by its contribution at primes $p \sim \frac{2\pi|\rho k|}{e}$. This is the rate at which the prime contributions of zeros grow with their height up the critical line.

With this analysis, it is time to get to the main theorem about the Riemann Hypothesis.

12.4.2. Main Theorem (A Möbius Parametrization of the Truncated Zero Condition)

The Parametrization Theorem

Theorem (Möbius parametrization of double-root configurations).

Let $H_0, H_1 \in \mathbb{C}$ *with* $H_1 \neq 0$. *The quadratic* $H_1 \cdot \rho^2 + (1 - H_1) \cdot \rho + H_0 = 0$ *has a double root if and only if*

$$(1 - H_1)^2 = 4 \cdot H_0 \cdot H_1$$

When this holds, the double root is

$$\rho = \frac{1}{2} - \frac{1}{2H_1},$$

with inverse parametrization

$$H_1 = \frac{1}{1-2\rho},\ H_0 = \frac{\rho^2}{1-2\rho}.$$

Overview

This is a step by step, the symbolic derivation that led from the Bernoulli form of the wall series to a clean Möbius parametrization of the truncated zero condition. Each step is recorded with its symbolic input and output, so that the chain of determinations is fully traceable. No numerical substitution is used in the derivation itself; the critical line emerges as a geometric consequence rather than as an a priori assumption.

STEP 1: The wall series and its low-order truncation.

Starting from Jensen's identity, the nontrivial-zero condition (asymptotically) reads

$$\frac{\rho}{\rho-1} = \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} \Gamma(1-\rho)}{\Gamma(-\rho-2n+2)(2n)!}. \tag{135}$$

Absorb the leading $\frac{\rho}{\rho-1}$ as a phantom $n=0$ entry and split each $B_{2n}\cdot(2n)!$ via the Wilson-filtered Von Staudt-Clausen identity:

$$B_{\{2n\}}\cdot(2n)! = N_n + \frac{\chi_{P(2n+1)}}{2n+1},\ N_n \in \mathbb{Z},\quad \chi_{P(m)} \in \{0,1\}. \tag{136}$$

Writing $X_n := \chi_{P(2n+1)}$, and truncating at $n=1$, the coefficients collapse beautifully because

$$\frac{2^{2n}}{((2n)!)^2} = 1 \text{ and } 2^{2n}\cdot\frac{2n+1}{[(2n)!\cdot(2n+1)!]} = 1 \text{ for } n = 0,1. \tag{137}$$

All prefactors equal 1 at low order, so the truncated zero condition takes the form of a quadratic in ρ:

$$(N_1 + X_1)\cdot\rho^2 + (1 - N_1 - X_1)\cdot\rho + (N_0 + X_0) = 0. \tag{138}$$

STEP 2: The actual arithmetic values at $n=0$ and $n=1$.

Computing the Wilson-filtered Von Staudt-Clausen values directly:

n	$2n+1$	prime?	$B_{\{2n\}}\cdot(2n)!$	N_n	X_n	Wilson mechanism
0	1	no	$B_0\cdot 0! = 1$	1	0	phantom: leading $\rho/(\rho-1)$
1	3	yes	$B_2\cdot 2! = 1/3$	0	1	Wilson: $2! \equiv -1 \pmod 3$

The complementarity is striking: at $n=0$ the entire contribution lies in the integer part; at $n=1$ it lies entirely in the prime part. The relevant sums for the quadratic are

$$N_0+X_0=1 \text{ and } N_1+X_1=1 \text{ (Wilson-VSCvalues).} \tag{139}$$

STEP 3: The quadratic at Wilson values

Substituting the Wilson-VSC values into the truncated quadratic gives

$$1\cdot\rho^2+0\cdot\rho+1=0, \text{ i.e., } \rho^2+1=0, \tag{140}$$

with roots $\rho=\pm i$. These are off the critical line by 1/2 in the real direction. So, the *low-order arithmetic truncation does not reproduce nontrivial zeros of ζ*. The information it carries is structural, not numerical.

STEP 4: Generalizing: replace (N_0+X_0, N_1+X_1) by free parameters (H_0, H_1)

Abandon the Wilson constraint and treat $H_0=N_0+X_0$, and $H_1=N_1+X_1$ as free complex parameters. The truncated quadratic becomes

$$H_1\rho^2+\left(1-H_1\rho\right)+H_0=0 \tag{141}$$

Under this generalization the truncated equation is a two-parameter family. The question becomes: for which free parameters (H_0,H_1) does this quadratic have a root at a prescribed ρ, and in particular on the critical line?

STEP 5: Impose a double-root condition: discriminant = 0

Require the most degenerate configuration, a single double root. The discriminant of the truncated quadratic is

$$\Delta=\left(1-H_1\right)^2-4\cdot H_0\cdot H_1 \tag{142}$$

Setting $\Delta=0$ yields the curve

$$\left(1-H_1\right)^2=4\cdot H_0\cdot H_1 \tag{143}$$

in the complex parameter plane. This is the locus of all (H_0,H_1) producing a double root.

STEP 6: Solve for the double-root location ρ

When $\Delta=0$, the double root is at

$$\rho=-\frac{B}{2A}=-\frac{\left(1-H_1\right)}{2H_1}=\frac{H_1-1}{2H_1} \tag{144}$$

This gives the forward Möbius map

$$\rho=\frac{1}{2}-\frac{1}{2H_1} \tag{145}$$

and its inverse:

$$H_1=\frac{1}{1-2\rho},\quad H_0=\frac{\rho^2}{1-2\rho} \tag{146}$$

STEP 7: The critical-line slice

From $\rho=\frac{1}{2}-\frac{1}{2H_1}$, the real part of ρ is

$$\operatorname{Re}(\rho)=\frac{1}{2}-\left(\frac{1}{2}\right)\cdot\operatorname{Re}\left(\frac{1}{H_1}\right). \tag{147}$$

So $\operatorname{Re}(\rho)=\frac{1}{2}$ if and only if $\operatorname{Re}\left(\frac{1}{H_1}\right)=0$, equivalently:

$$\operatorname{Re}(\rho)=\frac{1}{2}\Leftrightarrow H_1\in i\mathbb{R}\ \left(H_1 \text{ purely imaginary}\right). \tag{148}$$

The critical-line condition is therefore a one-real-codimension slice of the parameter space—purely geometric, not arithmetic. It emerges without being assumed.

Explicitly, writing $H_1=\frac{i}{2T}$ for real $T\neq 0$:

$$\rho=\frac{1}{2}+iT,\ H_0=-\frac{1}{2}+i\cdot\frac{1-4T^2}{8T}. \tag{149}$$

Both H_0 and H_1 are necessarily complex when $T\neq 0$. The calibration to a nontrivial critical-line root cannot be achieved with real parameters.

STEP 8: Locate the Wilson point in this geometry

The actual Wilson-VSC values are $(H_0,H_1)=(1,1)$. Evaluating the curve equation

$$(1-H_1)^2-4\cdot H_0\cdot H_1 \text{ at } (1,1): 0-4=-4. \tag{150}$$

The Wilson point lies off the double-root curve $\mathcal{C}$ at discrepancy −4. This number is exactly the discriminant of the truncated quadratic at Wilson values, consistent with its actual roots being $\pm i$ (a conjugate pair with squared distance 1 from the critical line).

Equivalently, the Wilson point sits on the level set $\{\Delta=-4\}$ of the discriminant, not on the zero-discriminant curve $\mathcal{C}$.

STEP 9: The X_0-N_1 relation under Wilson gauge

Fixing the Wilson gauge values $N_0=1$ and $X_1=1$, and demanding the truncated quadratic have a double root on the critical line, gives the algebraic relation between X_0 and N_1:

$$(N_1+1)^2-(4X_0+6)\cdot(N_1+1)+1=0 \tag{151}$$

or equivalently:

$$X_0=-\frac{3}{2}+\frac{N_1+1}{4}+\frac{1}{4[N_1+1]} \tag{152}$$

Setting

$$u=N_1+1=\frac{i}{2T}, \tag{153}$$

this becomes

$$u^2-(4X_0+6)u+1=0, \tag{154}$$

whose two roots are reciprocals (by Vieta: $u_1 \cdot u_2 = 1$). The two branches correspond to reciprocal T-values: if one root has $T = T_A$, the other has $T_B = -\frac{1}{4T_A}$.

The curve equation under Wilson gauge *equals the discriminant of the truncated quadratic*. They are the same polynomial. Hence, the $X_0 - N_1$ relation is literally the statement $\Delta = 0$.

Corollary 1: (Critical-line slice). *The double root lies on* $\mathrm{Re}(\rho) = 1/2$ *if and only if* $H_1 \in i\mathbb{R}$. *Explicitly,* $H_1 = \frac{i}{2T}$ *gives* $\rho = 1/2 + iT$.

Corollary 2 (Curve in ($\boldsymbol{H_0}$, $\boldsymbol{H_1}$)-space). *The double-root locus* $\mathcal{C} = \left\{(H_0, H_1) : (H_1 - 1)^2 = 4 \cdot H_0 \cdot H_1\right\}$ *is a smooth affine curve. The Möbius parametrization gives a bijection between* $\mathcal{C} \setminus \left\{(0,0)\right\}$ *and* $\mathbb{C} \setminus \{1/2\}$ *via the map* $(H_0, H_1) \leftrightarrow \rho$.

Corollary 3 (Wilson position). *The Wilson-VSC point* $(H_0, H_1) = (1,1)$ *does not lie on* $\mathcal{C}$; *the discrepancy* $(H_1 - 1)^2 - 4 \cdot H_0 \cdot H_1$ *at this point equals* −4, *exactly the discriminant of the truncated quadratic at Wilson values.*

Facts about the Curve

The double-root locus of the truncated zero condition

$$\boldsymbol{\mathcal{C}} := \left\{(H_0, H_1) \in \mathbb{C}^2 : (H_1 - 1)^2 = 4 \cdot H_0 \cdot H_1\right\}$$

Equivalently, is the locus where the truncated quadratic $H_1 \cdot \rho^2 + (1 - H_1) \cdot \rho + H_0 = 0$ has a double root.

Defining equation and degree

The curve is the affine complex algebraic curve defined by

$$F(H_0, H_1) := (H_1 - 1)^2 - 4 \cdot H_0 \cdot H_1 = 0.$$

Expanded as

$$H_1^2 - (4H_0 + 2) \cdot H_1 + 1 = 0.$$

It is degree 2 in H_1, degree 1 in H_0—a conic.

Smoothness: Computing partial derivatives:

$$\frac{\partial F}{\partial H^0} = -4H^1 \text{ and } \frac{\partial F}{\partial H^1} = 2(H^1 - 1) - 4H^0$$

The system $F = \partial F / \partial H_0 = \partial F / \partial H_1 = 0$ has no solutions in affine space. $\boldsymbol{\mathcal{C}}$ is smooth (non-singular) at every affine point. The curve is a smooth conic, $\boldsymbol{\mathcal{C}}$ has genus 0. It is birationally equivalent to $\mathbb{P}^1(\mathbb{C})$—the Riemann sphere. The map

$$\varphi : \mathbb{C} \setminus \{0\} \to \mathcal{C},\ H_1 \mapsto \left((H_1 - 1)^2 / (4H_1), H_1\right)$$

is a bijection away from the puncture at $H_1 = 0$. Projection to ρ gives the Möbius transformation:

$$\rho = (H_1 - 1)/(2H_1),\ H_1 = 1/(1 - 2\rho).$$

Points of $\boldsymbol{\mathcal{C}}$ with both $H_0, H_1 \in \mathbb{R}$ form the curve

$$H_0 = (H_1 - 1)^2 / (4H_1) \ (H_1 \in \mathbb{R} \setminus \{0\}).$$

This corresponds to ρ real. As H_1 ranges over $\mathbb{R}\setminus\{0\}$, the resulting ρ covers $\mathbb{R} \setminus \{1/2\}$.

Points of $\mathcal{C}$ with $\mathrm{Re}(\rho) = \frac{1}{2}$ are exactly $\{H_1 \in i\mathbb{R} \setminus \{0\}\}$.

On this slice:

$$\mathrm{Re}(H_0) = -1/2, \ \mathrm{Im}(H_0) = \frac{1-4T^2}{8T}, \ H_1 = \frac{i}{2T}.$$

The critical-line slice is a one-real-dimensional curve in $\mathcal{C}$, parametrized by $T \in \mathbb{R} \setminus \{0\}$. The real-point locus and the critical-line locus meet only at $H_1 = 0$ (point at infinity, $\rho \to 1/2$). Wilson-arithmetic-real configurations and critical-line configurations live in geometrically disjoint slices.

The map $\rho \mapsto 1 - \rho$ (pairing of zeros from ζ's functional equation) corresponds on $\mathcal{C}$ to:

$$H_1 \mapsto -H_1, \ H_0 \mapsto -(H_1 + 1)^2 / (4H_1).$$

This is an involution of $\mathcal{C}$ with fixed point only at $H_1 = 0$ (corresponding to $\rho = 1/2$). The map $\rho \mapsto \bar{\rho}$ corresponds to standard complex conjugation in the parameter:

$$H_1 \mapsto \overline{H_1}, \ H_0 \mapsto \overline{H_0}.$$

The composite $\rho \mapsto 1 - \bar{\rho}$—the Schwarz reflection across the critical line corresponds to

$$H_1 \mapsto -H_1.$$

The curve $\mathcal{C} = \left\{(H_1 - 1)^2 = 4H_0H_1\right\}$ is a smooth complex conic (genus 0) parametrized by a single complex variable via a Möbius map to ρ. Its symmetry group includes the involutions induced by ζ's functional equation and by complex conjugation.

The critical line $\mathrm{Re}(\rho) = 1/2$ is the fixed locus on of the Schwarz reflection $\rho \mapsto 1 - \bar{\rho}$, and equivalently, the imaginary-axis slice in the H_1 parameter. Real-arithmetic and critical-line configurations occupy disjoint slices, intersecting only at infinity ($\rho = 1/2$).

The geometric organization of $\mathcal{C}$ and its symmetries, its slices, the position of Wilson—encodes structural information about how the Bernoulli arithmetic of the wall series interacts with the symmetries of the Riemann zeta function. Whether the higher-order truncations preserve this structure under appropriate re-summation, with H_1 becoming purely imaginary at exactly the nontrivial zeros, is the concrete open question this geometry frames.

The critical line is not assumed. It is the fixed set of the Schwarz reflection on $\mathcal{C}$ geometrically determined by the symmetries of ζ.

The theorem establishes the following:

1) A clean symbolic parametrization of the $n = 0..1$ truncated zero condition by a single complex parameter H_1, via a Möbius map.

2) The critical-line condition $\mathrm{Re}(\rho) = \frac{1}{2}$ is identified with H_1 being purely imaginary, a geometric, not arithmetic, statement.

3) The Wilson-VSC arithmetic point (1, 1) sits at a specific computable discrepancy (−4) from the double-root curve $\mathcal{C}$.

4) Under Wilson gauge, the $X_0 - N_1$ relation is the discriminant of the truncated quadratic, exhibiting a reciprocal-root structure linking two T-branches.

12.4.3. Structural Remarks

1) The parametrization describes the structure of the truncated quadratic, not of the truncated wall series at the non-trivial roots. Wilson values produce truncated roots $\pm i$, and actual zeros of ζ, $\mathrm{Re}(\rho) = \frac{1}{2}$. The truncated quadratic captures the **leading structural skeleton** of the zero condition. Specifically:

The full zero condition

$$\frac{\rho}{\rho - 1} = \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} \Gamma(1-\rho)}{\Gamma(-\rho - 2n + 2)(2n)!} \tag{155}$$

is what defines nontrivial zeros.

2) The Möbius parametrization $\rho = \frac{1}{2} - \frac{1}{2H_1}$, arises directly from this equation at low order.

3) The critical-line condition $\mathrm{Re}(\rho) = \frac{1}{2} \Leftrightarrow H_1 \in i\mathbb{R}$ is a geometric consequence of the Möbius structure that appears already at low order.

4) Higher-order corrections from $n \geq 2n$ modify the values of H_0, H_1 but, if the structural Möbius form persists, would not break the critical-line geometry.

5) The critical-line condition $\sigma = \frac{1}{2}$ in the truncated parametrization is not an arbitrary algebraic feature. It is:

- The geometric image, under the Möbius map, of the imaginary-axis slice of the parameter H_1.
- A structural property of the zero condition itself, visible already at $n = 0..1$. linked to the wall series organization through the Wilson-VSC split.

6) The convergent prime sum $T_{\text{prime}}(s)$ is an entire function of s. In contrast, the zero equation $\rho/(\rho - 1)$ has a pole at $\rho = 1$.

7) T_{prime} is defined for all complex s, not just at zeros. It does not, by itself, detect zeros: $T_{\text{prime}}(\sigma + iT)$ varies smoothly with σ, T, and off the critical line gives values comparable to those at zeros of similar T.

8) The equation $T_{\text{int}}(\rho) = \rho/(\rho - 1) - T_{\text{prime}}(\rho)$ at zero asserts that the divergent integer-indexed part of the wall is constrained to equal an explicit convergent quantity. A rigorous interpretation of this asymptotic equality for instance via Borel summation or optimal truncation could extract information about the dis-

tribution of zeros.

9) The prime sum $T_{\text{prime}}(s)$ is reminiscent of the Kanemitsu-Kuzumaki exponential series and of related "Lucas-type" series over primes. The precise relation to Dirichlet series (via Mellin transform of T_{prime}) may merit further investigation.

12.4.4. Caveats and Open Questions

1) The splitting (118) is a formal decomposition of an asymptotic series. The original wall series (113) does not converge, and while the prime-part T_{prime} is convergent, the integer-part T_{int} remains divergent. The identity $\frac{\rho}{\rho-1}=T_{\text{int}}(\rho)+T_{\text{prime}}(\rho)$ inherits this asymptotic character.

2) $T_{\text{prime}}(s)$ is a rigorously defined entire function; its evaluation at a zero yields a specific complex value. But this value alone does not constrain ρ to the critical line: T_{prime} is an entire function, taking all complex values on preimages of any given value.

3) The result requires the truncated series. It establishes a structural connection linking two Bernoulli-based objects, the prime-counting formula (95) and the wall series (113), that had been studied separately in this work. Whether this connection yields analytical leverage is an open question.

4) One potential direction for further analysis: combine $T_{\text{prime}}(s)$ with the Perron-P identity of Section 10.2 (Equation (62) and Equation (73)).

$$\psi_0(1-ix)=(1-ix)+i\sum_{\rho}\frac{\mathrm{e}^{\rho P(x,F(\rho))}}{\rho}-\log(2\pi)-1/2\log\left(1-(1-ix)^{-2}\right). \tag{156}$$

$$\sum_{\rho}\frac{\mathrm{e}^{-\rho P(1/(ix),F(\rho))}}{\rho}=-i\left[\frac{x}{x-1}-\psi_0\left(\frac{x}{x-1}\right)-\log(2\pi)-\frac{1}{2}\log\left(\frac{2x-1}{x^2}\right)\right] \tag{157}$$

Both involve sums over primes or zeros; a joint analysis might produce new constraints.

12.5. Summary

This section has established:

1) Theorem 12: an exact Bernoulli-based prime counting formula, obtained via Von Staudt-Clausen + Wilson.

2) Lemma 13: the decomposition of $B_{2n}(2n)!$ into an integer part and a prime-detector term.

3) The convergent prime sum $T_{\text{prime}}(s)$ as an entire function of *s*.

4) The splitting of the wall series into a divergent integer-part and a convergent prime-part.

The bridge is genuine and new in this form. Its immediate use is structural: it demonstrates that the prime content of the Bernoulli numbers (Wilson/Von Staudt-Clausen) and their analytic content (ζ-values, wall series) are both present in the paper's derivation and can be disentangled by the splitting (117). Further work can be read through the provided references for study [6]-[11].

13. Conclusions

Fubini's theorem states that for any integral of any integrable function in $\mathbb{R}^{r+s}$ the interchange of order of integration is allowed. Tonelli's Theorem gives a converse for nonnegative functions. By strictly convergent manipulations from Jensen's integral representation, a self-referential integral Equation (15) that biconditionally characterizes the F-images of the nontrivial zeros of ζ. The equation admits a simplified form (18) in terms of classical $(x \pm i)^{-\rho}$ integrals, and reduces further via Laplace transforms to a Dirichlet-like series of incomplete gamma functions (27), (29). The Riemann Hypothesis is equivalent to the statement that every solution of these equations lies on the unit circle $|F| = 1$.

The primary contribution of this work is the derivation chain itself, which avoids the divergent-series and Fubini-interchange difficulties that have obstructed earlier algebraic attempts on RH via similar starting points.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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