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A Sigma-Based Detection Framework for Prime Constellations with Applications to Bounded Prime Gaps

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Abstract

A family of prime constellation detectors $\mathrm{D}(p,N)$ expressed as rational powers of 2π, built from the sum-of-divisors function σ. The central object $\mathrm{D}(p,N)=(2\pi)^{p+N+1-\left[\sigma(p)+\sigma(p+2N)\right]/2}$ equals 1 if and only if both p and $p+2N$ are prime, and is strictly less than 1 otherwise. This construction generalizes naturally to k-tuples via a product formula. A zeta-regularized product identity $\prod \mathrm{D}(p)=(2\pi)^{\frac{3}{8}}$ is created and decomposed into an associated sum of three explicitly characterized sub-series whose regularized total equals −5/48, a value determined purely by $\zeta(0)$ and $\zeta(-1)$. It is shown that the composite sub-series $F(1)$ converges to a constant numerically close to $\log_{10}2$. Zhang's bounded-gap theorem is recast in this form, showing that the average sieve sum $T(x,M)>1$ for $M\geq 6$, and extend the framework to k-prime clusters. The triple prime case $D_3(p) = 1$ is proven to have exactly one solution ($p = 3$) via a complete mod-3 obstruction, validating the framework in a case where the answer is known.

Keywords

Twin Primes, Prime Constellation in p, $p + 2N$, Gauss Multiplication Formula, Gamma Function, Sum-of-Divisors, Sieve Theory, Zhang, Bounded Primes, Prime Detecstor Function, Dirichlet Series, Bombieri-Vinogradov, Hardy-Littlewood Conjecture

1. Introduction

In prior papers [1], the author proposed methods that can possibly lead to a proof

of the twin prime conjecture by introducing the twin prime detector function, $\mathrm{D}(p)$, and its simplification, establishing the exact σ-characterisation of twin primes. The function $\mathrm{D}(p)$ is established using the Gauss Multiplication formula (GMF) in [2] and the Wallis product formula [3] [4]. A remainder bound for the counting of twin primes is derived and the double pole of the GMF-weighted Dirichlet series $\mathcal{L}(s)$ is discussed with its significance. This is a result of extensive work on prime detectors demonstrated in [1]. In this paper, a new detection function of prime constellations pairs, triples, and k-tuples of primes satisfying fixed gap conditions is created as a roadmap to some of the oldest problems in number theory. Wilson's theorem characterizes primes via a factorial congruence, but its computational complexity grows rapidly and it does not directly encode the structure of prime gaps. This paper introduces a unified, σ-based detection function using the same framework in [1] to encode the primality of both p and $p+2N$ into a single real number $\mathrm{D}(p,N)$, a power of 2π, that equals exactly 1 when both are prime and decays exponentially otherwise. The construction arises naturally from the Gauss multiplication formula (GMF) [2] also in [3] using the Wallis product in [4], for $\frac{\pi}{2}$ via for the Gamma function, and connect the detector to the Riemann zeta functional equation through a π-transformation in [1].

The paper is organized as follows. Section 2 defines $\mathrm{D}(p,N)$ and establishes its fundamental properties. Section 3 derives the zeta-regularized product and the three-series decomposition. Section 4 proves convergence of the composite subseries as new constant. Section 5 connects the $\mathrm{D}(p,N)$ frame work to Zhang's theorem [5] and the Maynard-Tao [6] extension. Section 6 treats the triple prime case as a validation. Section 7 discusses implications and open problems. The Hardy-Littlewood conjectures [7] predict the density of every admissible prime constellation. The proofs remained elusive until the Polymath collaboration [8].

2. The Detection Function $\mathrm{D}(p,N)$ See [1]

2.1. Definition

$\sigma(n)=\sum_{d|n} d$ denote the sum of divisors of n. For integers $p\geq 2$ and $N\geq 1$, define

$$\mathrm{D}(p,N)=(2\pi)^{p+N+1-\left[\sigma(p)+\sigma(p+2N)\right]/2} \tag{1}$$

When $N=1$, $\mathrm{D}(p,N)=\mathrm{D}(p,1)$ for the twin prime detector (1).

2.2. Fundamental Property

Theorem 1.

$\mathrm{D}(p,N)=1$ *if and only if both* p *and* $p+2N$ *are prime. For all other integer values of* $p\geq 2$ *and* $N\geq 1$, $\mathrm{D}(p,N)<1$.

Proof.

For any integer n: $\sigma(n)=n+1$ if and only if n is prime (since the only di-

visors of a prime are 1 and itself). If both p and $p+2N$ are prime, then $\sigma(p)=p+1$ and $\sigma(p+2N)=p+2N+1$, giving exponent $p+N+1-(p+1+p+2N+1)/2=0$. If either is composite, its σ value exceeds n + 1, making the exponent negative. Since 2π > 1, this gives $\mathrm{D}(p,N)<1$.The gap between $\mathrm{D}=1$ (prime pair) and $\mathrm{D}\le(2\pi)^{-\frac{3}{2}}\approx 0.063$ (smallest composite case, $p+2N$ a prime square) is nearly 0.937. The function does not approach 1 continuously. It *jumps discontinuously* to exactly 1 at prime pairs. The gap between $\mathrm{D}=1$ (prime pair) and $\mathrm{D}\le(2\pi)^{-\frac{3}{2}}\approx 0.063$ (smallest composite case, $p+2N$ a prime square) is nearly 0.937. The function does not approach 1 continuously. It *jumps discontinuously* to exactly 1 at prime pairs. Use the Gauss Multiplication Formula:

$$\prod_{k=0}^{p-1}\Gamma\left(y+\frac{k}{p}\right)=(2\pi)^{\frac{p-1}{2}}p^{\left(\frac{1}{2}-p\cdot y\right)}\Gamma(p\cdot y) \tag{2}$$

Define the single prime detector $\mathrm{M}(p)$. Let $\mathrm{GMF}(p)$ be the Gauss Multiplication Formula:

$$\mathrm{GMF}(p)=(2\pi)^{\frac{p-1}{2}}p^{\left(\frac{1}{2}-p\cdot y\right)}\frac{\Gamma(p\cdot y)}{\prod_{k=0}^{p-1}\Gamma\left(y+\frac{k}{p}\right)}=1 \tag{3}$$

Let $y=1$, then, the relation (16) can now be written as

$$(2\pi)^{\frac{1-p}{2}}p^{p-\frac{1}{2}}\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)=\Gamma(p) \tag{4}$$

Let $p=\dfrac{\sigma(p)}{y}$, then, the relation (4) can now be written as

$$(2\pi)^{\frac{1-\sigma(p)}{2}}\sigma(p)^{p-\frac{1}{2}}\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)=\Gamma(p) \tag{5}$$

in invariant to the substitution $\sigma(p)\Leftrightarrow p$, then,

$$(2\pi)^{\frac{1-p}{2}}p^{p-\frac{1}{2}}\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)=(2\pi)^{\frac{1-\sigma(p)}{2}}\sigma(p)^{p-\frac{1}{2}}\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right) \tag{6}$$

Let a prime detection function be $\mathrm{M}(p)$. Then, from (6) all primes p, satisfy the single prime detector:

$$\mathrm{M}(p)=(2\pi)^{\frac{\sigma(p)-p}{2}}\left(\frac{p}{\sigma(p)}\right)^{p-\frac{1}{2}}\frac{\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)}{\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)}=1 \tag{7}$$

Since, for all primes, *p*, $\sigma(p)-p=1$, then, from (7), for all primes, *p,*

$$\mathrm{M}(p)=\sqrt{2\pi}\frac{p^{p-\frac{1}{2}}\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)}{\sigma(p)^{p-\frac{1}{2}}\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)}\begin{cases}=1 & \text{if } \{p,p+2\}\in\text{primes}\\ \neq 1 & \text{otherwise}\end{cases} \tag{8}$$

It follows that for a prime $p+2N$,

$$\mathrm{M}(p+2N)=\sqrt{2\pi}\frac{p^{p+2N-\frac{1}{2}}\prod_{k=0}^{p+2n-1}\Gamma\left(\frac{p+2n+k}{p+2n}\right)}{\sigma(p+2n)^{p+2n-\frac{1}{2}}\prod_{k=0}^{\sigma(p+2n)-1}\Gamma\left(\frac{p+2n+k}{\sigma(p+2n)}\right)} \tag{9}$$

$$\begin{cases}=1 & \text{if } \{p+2N\}\in \text{primes}\\ \neq 1 & \text{otherwise}\end{cases}$$

The paired prime detector $\mathrm{D}(p,p+2N)$ for paired prime of $\{p,p+2N\}$, can now be written as the product of (8) and (9). Hence,

$$\begin{aligned}&\mathrm{D}(p,p+2N)\\&=\left\{\sqrt{2\pi}\frac{p^{p+2N-\frac{1}{2}}\prod_{k=0}^{p+2N-1}\Gamma\left(\frac{p+2N+k}{p+2N}\right)}{\sigma(p+2N)^{p+2N-\frac{1}{2}}\prod_{k=0}^{\sigma(p+2N)-1}\Gamma\left(\frac{p+2N+k}{\sigma(p+2N)}\right)}\right\}\\&\cdot\left\{\sqrt{2\pi}\frac{p^{p-\frac{1}{2}}\prod_{k=0}^{p-1}\Gamma\left(\frac{p+k}{p}\right)}{\sigma(p)^{\sigma(p)-\frac{1}{2}}\prod_{k=0}^{\sigma(p)-1}\Gamma\left(\frac{p+k}{\sigma(p)}\right)}\right\}\end{aligned} \tag{10}$$

$$\begin{cases}=1 & \text{if } \{p,p+2N\}\in \text{primes}\\ \neq 1 & \text{otherwise}\end{cases}$$

Using the standard Gauss identity:

$$\prod_{k=0}^{p-1}\Gamma\left(z+\frac{k}{p}\right)=(2\pi)^{\frac{p-1}{2}}p^{\left(\frac{1}{2}-pz\right)}\Gamma(pz) \tag{11}$$

And, apply the Gauss multiplication theorem [3] to every product:

Put $z=1,n=p+2N$ in the numerator of the first bracket, and put $z=\dfrac{p+2N}{\sigma(p+2N)}$, $n=\sigma(p+2N)$ in the denominator of the first bracket.

Put $z=1,n=p$ in the numerator of the second bracket, and put $z=\dfrac{p}{\sigma(p+2N)}$, $n=\sigma(p)$ in the denominator of the second bracket.

We get:

$$\begin{aligned}&\mathrm{D}(p,p+2N)\\&=\left\{\sqrt{2\pi}\frac{(p+2N)^{p+2N-\frac{1}{2}}\cdot(2\pi)^{\frac{p+2N-1}{2}}(p+2N)^{\left(\frac{1}{2}-p-2N\right)}\Gamma(p+2N)}{\sigma(p+2N)^{p+2N-\frac{1}{2}}\cdot(2\pi)^{\frac{\sigma(p+2N)-1}{2}}(\sigma(p+2N))^{\left(\frac{1}{2}-p-2N\right)}\Gamma(p+2N)}\right\}\\&\cdot\left\{\sqrt{2\pi}\frac{p^{p-\frac{1}{2}}\cdot(2\pi)^{\frac{p-1}{2}}(p)^{\left(\frac{1}{2}-p\right)}\Gamma(p)}{\sigma(p)^{\sigma(p)-\frac{1}{2}}\cdot(2\pi)^{\frac{\sigma(p)-1}{2}}(\sigma(p))^{\left(\frac{1}{2}-\sigma(p)\right)}\Gamma(p)}\right\}\end{aligned}$$

$$\mathrm{D}(p,1)=(2\pi)^{p+N+1-\frac{\sigma(p+2N)+\sigma(p)-1}{2}} \tag{12}$$

If both p and $p+2N$ are primes then, $\sigma(p)=p+1$ and $\sigma(p+2N)=p+2N+1$ and

$$\mathrm{D}(p,1)\begin{cases}=1 & \text{if } \{p,p+2\mathrm{N}\}\in \text{primes}\\ \neq 1 & \text{otherwise}\end{cases} \tag{13}$$

2.3. Derivation of $\mathrm{D}(p,1)$ from the Wallis Product

The construction can also be derived using the GMP and the Wallis product for $\frac{\pi}{2}$:

$$\frac{\pi}{2}=\prod_{k=1}^{\infty}\left(\frac{2k}{2k-1}\right)\cdot\left(\frac{2k}{2k+1}\right) \tag{14}$$

$$\prod_{k=0}^{n-1}\Gamma\left(y+\frac{k}{n}\right)=(2\pi)^{\frac{n-1}{2}}n^{\left(\frac{1}{2}-n\cdot y\right)}\Gamma(n\cdot y) \tag{15}$$

to each block, with $n=p$ and $n=\sigma(p)$ in the denominator, $n=p+2N$ and $n=\sigma(p+2N)$ yields the simplification. All $\Gamma(p)$ and $\Gamma(p+2N)$ factors cancel, and the prefactor $2\sqrt{2}\cdot\pi^{3/2}=\frac{(2\pi)^{\left\{\frac{3}{2}\right\}}}{\sqrt{2\pi}}$ collapses, leaving Equation (1).

3. Zeta-Regularized Product and the Three-Series Decomposition

3.1. The Regularized Product

Theorem 2.

Define the twin-prime detector exponent

$$\mathrm{B}(n)=n+2-\frac{\sigma(n)+\sigma(n+2)}{2}$$

Where $\mathrm{D}(n,1)=(2\pi)^{\mathrm{B}(n)}$ is the twin prime detector of Theorem 1.

Then under zeta regularization,

$$\sum_{\{n\geq 2\}}\operatorname{reg}B(n)=\frac{3}{8}$$

and consequently,

$$\prod_{n=2}^{\operatorname{reg}}\mathrm{D}(n,1)=(2\pi)^{\frac{3}{8}}$$

Proof.

Using Theorem 1, write the exponent sum of $\mathrm{D}(n,1)$ as:

$$\sum_{\{n\geq 2\}}B(n)=\sum_{\{n\geq 2\}}(n+2)-\frac{1}{2}\sum_{\{n\geq 2\}}\left(\sigma(n+2)\right)-\frac{1}{2}\sum_{\{n\geq 2\}}\left(\sigma(n)\right) \tag{16}$$

Using the Dirichlet series

$$\sum_{n=1}^{\infty}\frac{\sigma(n)}{n^{s}}=\zeta(s)\zeta(s-1),\ \Re(s)>2. \tag{17}$$

The Abel (or exponential) regularization gives:

$$\lim_{x\to 0}\sum_{n\geq 1}^{\infty}\sigma(n)\mathrm{e}^{-nx}=\text{Finite Part}\left\{\lim_{x\to 0}\sum_{m=1}^{\infty}\frac{m}{\mathrm{e}^{mx}-1}\right\}=\frac{1}{24} \tag{18}$$

and

$$\sum_{n\geq 1}^{\infty}\sigma(n+k)=\frac{1}{24}-\sum_{m=1}^{k}\sigma(m)\ \ (\text{regularized}) \tag{19}$$

$$\sum_{n\geq 2}^{\infty}(n+2)=\sum_{n=1}^{\infty}(n)+\sum_{n=1}^{\infty}(1)=-\frac{49}{12} \tag{20}$$

$$\boxed{\sum_{n\geq 2}^{\infty}(n+2)=\zeta(-1)+2\zeta(0)=-\frac{13}{12}+2\left(-\frac{3}{2}\right)=-\frac{49}{12}} \tag{21}$$

$$\sum_{n\geq 2}^{\infty}\sigma(n+2)=\sum_{n=4}^{\infty}\sigma(n)-\sigma(1)-\sigma(2)-\sigma(3)$$

$$\boxed{\sum_{n\geq 2}^{\infty}\sigma(n+2)=\frac{1}{24}-1-3-4=-\frac{191}{24}} \tag{22}$$

$$\sum_{n\geq 2}^{\infty}(\sigma(n))=\sum_{n=1}^{\infty}\sigma(n)-\sigma(1)=\frac{1}{24}-1=-\frac{23}{24}$$

The total is:

$$\boxed{\sum_{n\geq 2}B(n)=-\frac{49}{12}+\frac{191}{48}+\frac{23}{48}=\frac{18}{48}=\frac{3}{8}} \tag{23}$$

Every twin prime pair, $n,n+2$, contributes $\mathrm{D}(n,1)=1$, at twin prime pairs leaving the product invariant. The value $\frac{3}{8}$ is determined entirely by the non-twin terms and is independent of how many twin prime-pairs exist. All divergences cancel exactly, leaving a finite rational number. That's a strong signal the construction is a balanced regularized combination, the kind that often shows up in modular/zeta structures.

Remarks

1. Role of twin primes in the regularized sum. At every twin-prime pair $(p,p+2)$, $B(p)=0$ and $D(p,1)=1$. These terms contribute zero to the sum $\sum B(n)$ term-by-term, which is why the product $\prod_{n=2}^{\text{reg}}\mathrm{D}(n,1)=(2\pi)^{\frac{3}{8}}$ is "invariant" under the inclusion or exclusion of twin primes. However, this does not mean the regularized value $\frac{3}{8}$ is independent of the distribution of twin primes in a deep sense. It means only that twin primes contribute zero as individual terms. The density structure of composites, isolated primes, and twin primes in $\mathbb{N}$ is encoded implicitly through the zeta-regularized sums, since those regularizations treat $\mathbb{N}$ as a whole and do not "see" the partition.

2. Interpretation of zeta regularization here. The sums $\sum_{n\geq 2}^{\text{reg}}$ are defined by Dirichlet-series continuation: for example, $\sum_{n\geq 2}^{\text{reg}}\sigma(n)=\lim_{s\to 0}\zeta(s)\zeta(s-1)$ via

the Dirichlet series $\sum \sigma(n) n^{-s}$. This is consistent with the classical Abel-regularized value (the finite part at $x = 0^+$ of $\sum \sigma(n) \mathrm{e}^{-nx}$), which we derived in Theorem 3 as $\frac{1}{24}$. The two regularization schemes agree on this example; the present section uses the Dirichlet-series version for algebraic convenience.

3.2. The Three-Series Decomposition

Define the Mangold function:

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \\ 0 & \text{otherwise} \end{cases}$$

It satisfies:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \quad (\Re(s) > 1),$$

Define the correlation function

$$C_{\text{Twin}}(x) = \sum_{n \geq 1} \Lambda(n) \Lambda(n+2N) \mathrm{e}^{-nx}$$

This is the analytic proxy for n *and* $n+2$ are both prime. Then, as $x \to 0^+$, $C_{\text{Twin}}(x)$ gives the twin prime density. By analytic continuation:

$$\sum_{n=1}^{\infty} \left(n - \frac{\sigma(n)}{2} \right) \underset{reg}{=} -\frac{5}{48}$$

Let

$$F(x) = \sum_{n \geq 1}^{\infty} \left(n - \frac{\sigma(n)}{2} \right) \mathrm{e}^{-nx}$$

Theorem 3 (Regularized Arithmetic Decomposition of $n - \frac{\sigma(n)}{n}$)

Define the smoother series

$$F(x) = \sum_{n=1}^{\infty} \left(n - \frac{\sigma(n)}{2} \right) \mathrm{e}^{-nx}$$

Then, as $x \to 0^+$, the function admits asymptotic expansion

$$F(x) = \left(1 - \frac{\pi^2}{12} \right) \frac{1}{x^2} + \frac{1}{4x} - \frac{5}{48} + O(x^2)$$

In particular the finite part satisfied

$$\lim_{x \to 0^+} \sum_{n \geq 1}^{\infty} \left(n - \frac{\sigma(n)}{2} \right) \mathrm{e}^{-nx} = -\frac{5}{48}$$

Proof:

Split $F(x)$: $F(x) = \sum_{n=1}^{\infty} \left(n - \frac{\sigma(n)}{2} \right) \mathrm{e}^{-nx} = A(x) + B(x)$:

$$A(x)=\sum_{n\geq 1} n\mathrm{e}^{-nx},\ B(x)=\frac{1}{2}\sum_{n\geq 1}\sigma(n)\mathrm{e}^{-nx}.$$

The closed form of $A(x)$ is:

$$A(x)=\frac{\mathrm{e}^{-x}}{\left(1-\mathrm{e}^{-x}\right)^2}$$

Expanding at $x\to 0^+$,

$$A(x)=\frac{1}{x^2}-\frac{1}{12}+\frac{x^2}{240}+O\left(x^4\right)$$

Use the Lambert series identity:

$$B(x)=\frac{1}{2}\sum_{m\geq 1}\frac{m}{\mathrm{e}^{mx}-1}=\frac{\pi^2}{12x^2}-\frac{1}{4x}+\frac{1}{48}+O\left(x^2\right)$$

Subtracting,

$$F(x)=\left(1-\frac{\pi^2}{12}\right)\frac{1}{x^2}+\frac{1}{4x}-\frac{5}{48}+O\left(x^2\right)$$

$F(x)$ has a small-x asymptotic expansion whose finite part equals $-\frac{5}{48}$.

Definition: Partition the integers into three disjoint sets:

$$\mathbb{N}=I_{TP}\cup I_{NTP}\cup I_{\mathrm{comp}}$$

where

I_{TP} is the set of isolated twin prime indices;

I_{NTP} is the set of isolated prime indices not in I_{TP};

I_{comp} is the set of isolated composite indices.

Definition: For each index set I_x, define the partial generating function

$$F_i(x)=\sum_{n\in I_i}^{\infty} A(n)\mathrm{e}^{-nx},\ A(n)=n-\frac{\sigma(n)}{2},\ x>0$$

Each $F_i(x)$ converges absolutely for $x>0$.

If $F_i(x)$ admits an asymptotic expansion

$$F_i(x)=\frac{\alpha_i}{x^2}+\frac{\beta_i}{x}+Q_i+O(x)\ \text{as}\ x\to 0^+$$

Then the scalar Q_i is called the finite part of $F_i(x)$.

Theorem 4. (Decomposition of the finite part (conjectural))

Assuming the Hardy-Littlewood conjecture, each $F_i(x)$ *admits an asymptotic expansion of the stated form, and the finite parts satisfy*:

$$Q_{TP}+Q_{NTP}+Q_{composite}=-\frac{5}{48}$$

The proof follows from Theorem 3.

Note: The partial sums diverge classically; under the balanced regularized combination, all divergences cancel exactly, leaving a finite rational number. That's a strong signal that the construction is a balanced regularized combination, the kind that often shows up in modular/zeta structures and the identity holds only under

zeta regularization.

4. Convergence of the Composite Series

4.1. Powers of 2 Contribute Exactly 1/2

Lemma 1.

For every $k \geq 1$, $A\left(2^k\right) = \frac{1}{2}$. *Consequently, the regularised sum over powers of* 2 *is* $\frac{1}{2}\left(\zeta(0) - 1\right) = -\frac{3}{4}$.

Proof.

$\sigma\left(2^k\right) = 2^{k+1} - 1$, so, $2^k - \frac{\sigma\left(2^k\right)}{2} = 2^k - \frac{2^{k+1} - 1}{2} = \frac{1}{2}$ for all k.

4.2. The Composite Series Converges

Theorem 5.

Let

$$B(n) = n + 2 - \frac{\sigma(n+2) + \sigma(n)}{2}, D(n,1) = (2\pi)^{B(n)}$$

Define

$$C = \sum_{n=1}^{\infty} \left[D(n,1) \cdot 1_{n,n+2 \text{ not twin prime}} \right]$$

Then the series defining C depends only on the non-twins and C converges absolutely.

To a constant numerically calculated to equal 0.30104… ≈ $\log_{10} 2$ = 0.30103….

Note: The numerical value 0.30104 agrees with $\log_{10} 2 = 0.30103$ to five significant figures; whether this equality is exact is an open question.

Proof sketch.

Let $e(m) := \sigma(m) - m - 1 \geq 0$, which is zero iff m is prime. Then $B(n) = -\frac{1}{2}\left[e(n) + e(n+2)\right]$. For $(n, n+2)$ *non-twin,* $e(n) + e(n+2) \geq 2$, so $\mathrm{D}(n,1) \leq (2\pi)^{-1}$. Grouping by excess level $e(n) + e(n+2) = 2j$ and bounding the number of terms at each level by $O(n)$, the series is dominated by $\sum_j C_j (2\pi)^{-1}$, which converges as a geometric series.

Note: Direct computation gives $C \approx 0.30104$ to five decimal places.

5. Connection to Bounded Prime Gaps

The $D(p,N)$ Sieve For a prime p, define the sieve sum over M gaps:

$$S(p,M) = \sum_{N=1}^{M} D(p,N) \geq \#\{N \leq M : p + 2N \text{ prime}\}$$

The inequality holds because each term is at most 1, equaling 1 precisely when $p + 2N$ is prime. The average sieve sum is

$$T(x,M)=\frac{1}{\pi(x)}\sum_{p\geq x}S(p,M)\{p\leq x\}$$

Computing $T(x,M)$ for $x=50000$ reveals the crossover point:

M	Max gap $2M$	$T(x, M)$	$T>1$?
5	10	0.872	no
6	12	1.145	YES
7	14	1.313	YES
10	20	1.907	YES
123	246	25.3	YES

5.1. Observation

Zhang's bounded-gap theorem [5] can be restated in the present notation: there are infinitely many primes p and integers $N\leq 35000000$ such that $\mathrm{D}(p,N)=1$. Numerical computation of $T(x,M)$ at $x=50000$ shows that the average sieve-sum exceeds 1 beginning at $M=6$. This is an empirical observation about the average behavior of $S(p,M)$ at this particular x; it does not, on its own, establish an asymptotic bounded-gap result, which requires deep distribution estimates for primes in arithmetic progressions as employed by Zhang [5], Maynard-Tao [6], and the Polymath collaboration [8] in 2014. The Polymath improvements reduce N to ≤123, and the Maynard-Tao machinery extends the result to arbitrary finite prime tuples. These are deep results, whose proofs rely on strong distribution estimates for primes in arithmetic progressions, sieve theory, and, in Zhang's case, exponential-sum bounds of Deligne-type; a rigorous proof is beyond the scope of the present framework. The D-formalism provides a compact restatement but does not, by itself, circumvent the deep analytic content of these theorems.

5.2. Quantitative Bound and Extension

Under GEH, Polymath (2014) [8], conditionally established gap ≤ 12; their unconditional bound is gap ≤ 246. The D-formalism offers a restatement of such results but does not provide independent access to them.

6. The Triple Prime Case: Validation

6.1. The Triple Detector $\mathrm{D}_3(p,p+2,p+4)$

For the constellation $(p,p+2,p+4)$, applying the Gauss formula to the three-block expression in the image yields

$$\mathrm{D}_3(p)=(2\pi)^{\left[3(p+3)-\sigma(p)-\sigma(p+2)-\sigma(p+4)\right]/2} \tag{24}$$

which equals 1 if and only if $p,p+2$, and $p+4$ are all prime.

6.2. Uniqueness of (3, 5, 7)

Theorem 8.

$D_3(p)=1$ *if and only if* $p=3$. *Consequently,* $(3,5,7)$ *is the only prime triple of the form* $(p, p+2, p+4)$.

Proof.

Among any three consecutive integers $\{p, p+2, p+4\}$, exactly one is divisible by 3 (since the residues mod 3 cycle through 0, 1, 2 and the set covers three consecutive even-spaced values). If $p \equiv 0 \pmod 3$, then $p=3$ is the only prime possibility. If $p \equiv 1 \pmod 3$, then $p+2 \equiv 0 \pmod 3$, so $p+2$ is composite for $p+2>3$. If $p \equiv 2 \pmod 3$, then $p+4 \equiv 0 \pmod 3$, so $p+4$ is composite for $p+4>3$. Only $p=3$ satisfies all three simultaneously.

This result validates the entire framework: when a definitive arithmetic obstruction exists, the D-function correctly yields a finite, computable answer. The absence of any analogous obstruction for twin primes is precisely why the conjecture remains open.

7. Summary of Results

7.1. A Note on Regularized Identities (Formal Sums)

The series diverges in the classical sense. You assign a value using an external analytic method, e.g.: analytic continuation (ζ-regularization) Abel/Cesàro summation cutoff subtraction The "sum" is not a limit of partial sums, but a renormalized value. For example:

$1+2+3+4+\cdots=\zeta$ regularized to = −1/12.

Feature	Ordinary convergence	Regularization
Based on limit?	Yes	No
Unique value?	Yes	Depends on method
Physically meaningful?	Always	Often (physics), but formal
Algebra rules safe?	Yes (with care)	Must be handled carefully
Rearrangement allowed?	(conditionally tricky)	Generally justified

7.2. The $D(p,N)$ Framework

Achieves the following:

1) A universal prime pair detector valid for all gaps $2N$, expressed as a single power of 2π.

2) A zeta-regularized product identity $\prod D(p)=(2\pi)^{\frac{3}{8}}$ with closed-form exponent.

3) A three-series decomposition with regularized sum $-\frac{5}{48}$, determined purely by $\zeta(0)$ and $\zeta(-1)$.

4) Connection to Bounded Prime Gaps, Zhang's theorem and the Maynard–Tao extension in $D(p,N)$ language, with the crossover $T(x,M)>1$ at $M=6$.

5) (Polymath 2014, "Variants of the Selberg sieve, and bounded intervals containing many primes").

6) Complete proof that (3, 5, 7) is the unique prime triple, validating the framework.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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