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The Properties of Coarsely Quasihyperbolic Mappings in Metric Spaces

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Abstract

In this paper, we investigate the behavior of quasigeodesics and coarse quasihyperbolic lengths under coarsely quasihyperbolic mappings between domains in metric spaces. Using the distance distortion inequalities and coarse-length estimates, we show that such mappings and their inverses send quasigeodesic arcs to solid arcs with explicitly controlled parameters. Then we construct a homeomorphism of a one-dimensional interval which, together with its inverse, sends every quasigeodesic arc to a solid arc but is not coarsely quasihyperbolic. Finally, we establish a characterization of coarsely quasihyperbolic mappings in terms of two-sided coarse-length bounds for the images of short arcs.

Keywords

Coarsely Quasihyperbolic Mapping, Quasigeodesic, Solid Arc, Coarse Quasihyperbolic Length, Metric Space

1. Introduction

In 1976, in order to investigate the Riemann mapping theorem in higher-dimensional quasiconformal mappings, Gehring *et al.* in [1] first introduced the quasihyperbolic metric on Euclidean spaces. It has since been studied in Banach spaces and general metric spaces, and has become an important tool in quasiconformal geometry; for related research can be found in references [2] [3]. In the 1990s, based on the quasihyperbolic metric, Väisälä systematically investigated freely quasiconformal mappings, coarsely quasihyperbolic mappings, and solid arcs in Banach spaces, yielding a series of fundamental results on freely quasiconformal mappings and quasihyperbolic mappings, see [4]-[6]. As the setting was extended from Banach spaces to general metric spaces, the relationships among quasiconvexity, the path geometry of domains, and the quasihyperbolic distance have con-

tinued to attract attention. Related mapping properties in metric spaces have been investigated, among others, in [3] [7]-[10].

Let $f: G \to G'$ be a homeomorphism between domains. We examine two related questions: whether coarsely quasihyperbolic mappings send quasigeodesics to solid arcs, and whether the preservation of solid arcs alone implies coarse quasihyperbolicity. The first question admits a quantitative answer, whereas the second has a negative answer without an additional distancecontrol hypothesis. Section 3 proves the preservation result with explicit constants and provides a one-dimensional counterexample to its converse. Section 4 characterizes coarse quasihyperbolicity by two-sided bounds for the coarse quasihyperbolic lengths of images of short arcs. Our criterion is formulated as a length-control condition, rather than as preservation of solid arcs alone.

2. Preliminaries

Let X and Y be two metric spaces, and let $G \subsetneq X$ and $G' \subsetneq Y$ be nonempty domains. We write $|x-y|$ for $d_X(x,y)$ when $x, y \in X$, and use the analogous notation in Y. For $x \in G$, set

$$\delta_G(x) = \operatorname{dist}(x, X \setminus G).$$

We assume that $0 < \delta_G(x) < \infty$ for $x \in G$ and analogously $0 < \delta_{G'}(x') < \infty$ for $x' \in G'$. For a rectifiable arc $\gamma \subset G$, its quasihyperbolic length is

$$\ell_{k_G}(\gamma) = \int_\gamma \frac{\mathrm{d}s}{\delta_G(x)}.$$

The quasihyperbolic distance between $x, y \in G$ is

$$k_G(x,y) = \inf_\gamma \ell_{k_G}(\gamma)$$

where the infimum ranges over rectifiable arcs in G joining x to y. In what follows, the point pairs under consideration are assumed to have finite quasihyperbolic distance. For an arc γ and $a, b \in \gamma$, $\gamma[a,b]$ denotes the subarc joining a to b.

Definition 2.1 (Coarsely quasihyperbolic mapping)

For $M \geq 1$ and $C \geq 0$, a homeomorphism $f: G \to G'$ is called (M,C)-coarsely quasihyperbolic (abbreviated (M,C)-CQH), if for every $x, y \in G$,

$$\frac{k_G(x,y) - C}{M} \leq k_{G'}(f(x), f(y)) \leq M k_G(x,y) + C.$$

Definition 2.2 (Coarse quasihyperbolic length)

Let γ be an arc in a metric space X, and let $h \geq 0$. If a finite sequence of points $\overline{x} = (x_0, \cdots, x_n)(n \geq 1)$ ordered successively on γ satisfies: $k_G(x_{j-1}, x_j) \geq h$ for all $1 \leq j \leq n$, then $\overline{x}$ is called h-coarse sequences. Let $\Phi(\gamma, h)$ denote the family of all h-coarse sequences of γ. Define

$$s(\overline{x}) = \sum_{j=1}^{n} k_G(x_{j-1}, x_j) \quad \text{and} \quad \ell_{k_G}(\gamma, h) = \sup\{s(\overline{x}) : \overline{x} \in \Phi(\gamma, h)\},$$

where $\ell_{k_G}(\gamma,h)$ is called the h-coarse quasihyperbolic length of γ. We also adopt the convention that $\ell_{k_G}(\gamma,h)=0$ when $\Phi(\gamma,h)=\varnothing$. In particular,

$$\ell_{k_G}(\gamma,0)=\ell_{k_G}(\gamma).$$

Definition 2.3 (Solid arc)

Let $h\geq 0$ and $c\geq 1$. Let $\gamma\subset G$ be an arc. If for any $x,y\in\gamma$, we have

$$\ell_{k_G}(\gamma[x,y],h)\leq ck_G(x,y),$$

Then γ is called a (c,h)-solid arc. In particular, when $h=0$, γ is called a c-quasigeodesic (or c-neargeodesic).

Definition 2.4 (Short arc)

Let $x,y\in G$ and $L>0$. If a curve γ in G connecting x and y satisfies

$$\ell_{k_G}(\gamma)\leq k_G(x,y)+L,$$

then γ is called an L-short arc.

Remark 2.5

The criterion remains equivalent when one short curve is selected for each endpoint pair in each direction.

3. Invariance of Solid Arcs under Coarsely Quasihyperbolic Mappings

We first give quantitative bounds for the images of quasigeodesics.

Theorem 3.1

Suppose that $f:G\to G'$ is a (M,C)-coarsely quasihyperbolic mapping, where $M\geq 1$ and $C\geq 0$. If $\gamma\subset G$ is a c-quasigeodesic, $c\geq 1$, then $f(\gamma)\subset G'$ is a (c',h')-solid arc, where

$$c'=2cM(M+1),\qquad h'=cM(2C+1)+C.$$

The analogous assertion holds for $f^{-1}:G'\to G$.

Proof: Let $a,b\in\gamma$. Since γ is a c-quasigeodesic, we have

$$\ell_{k_G}(\gamma[a,b])\leq ck_G(a,b). \tag{3.1}$$

Thus, it suffices to prove that

$$\ell_{k_{G'}}(f(\gamma[a,b]),h')\leq c'k_{G'}(f(a),f(b))\cdot \tag{3.2}$$

We divide the proof into the following two cases:

$$\text{Case 1}: k_G(a,b)<2C+1.$$

For any $u,v\in\gamma[a,b]$, it follows from the quasihyperbolic distance and inequality (3.1) that

$$k_G(u,v)\leq \ell_{k_G}(\gamma[a,b])\leq ck_G(a,b)<c(2C+1).$$

Furthermore, by the definition of a (M,C)-coarsely quasihyperbolic mapping, we obtain

$$k_{G'}\left(f(u),f(v)\right)\le Mk_G(u,v)+C<cM(2C+1)+C=h'.$$

Therefore, $f\left(\gamma[a,b]\right)$ contains no h'-coarse sequences, so its h'-coarse length is zero. In this case, inequality (3.2) holds.

Case 2: $k_G(a,b)\ge 2C+1$.

Take an arbitrary h'-coarse sequence $(a_0',\cdots,a_n')$ on $f\left(\gamma[a,b]\right)$, and let $a_i=f^{-1}(a_i')$. Since $f:G\to G'$ is a homeomorphism and $\gamma\subset G$, its restriction $f|_{\gamma[a,b]}:\gamma[a,b]\to f\left(\gamma[a,b]\right)$ on the arc $\gamma[a,b]$ is also a homeomorphism. Consequently, the sequence of points $(a_0,\cdots,a_n)$ is also ordered successively along the arc $\gamma[a,b]$.

According to the definition of a (M,C)-coarsely quasihyperbolic mapping, for each i we have

$$k_G(a_{i-1},a_i)\ge\frac{k_{G'}(a_{i-1}',a_i')-C}{M}\ge\frac{h'-C}{M}=c(2C+1)\ge 2C+1. \tag{3.3}$$

Combining inequality (3.1), we get

$$\sum_{i=1}^{n}k_G(a_{i-1},a_i)\le\ell_{k_G}\left(\gamma[a,b]\right)\le ck_G(a,b).$$

Combining this with inequality (3.3), we can deduce

$$n\le\frac{ck_G(a,b)}{2C+1}. \tag{3.4}$$

Then, combining with inequality (3.4), we obtain

$$\begin{aligned}\sum_{i=1}^{n}k_{G'}(a_{i-1}',a_i')&\le M\sum_{i=1}^{n}k_G(a_{i-1},a_i)+nC\\&\le Mck_G(a,b)+\frac{cCk_G(a,b)}{2C+1}\\&\le c(M+1)k_G(a,b).\end{aligned}$$

Since the above estimate holds for any arbitrarily chosen h'-coarse sequence on the subarc $f\left(\gamma[a,b]\right)$, and $c(M+1)k_G(a,b)$ is independent of the choice of the sequence, this value is a common upper bound. Taking the supremum over all h'-coarse sequences, we obtain

$$\ell_{k_{G'}}\left(f\left(\gamma[a,b]\right),h'\right)\le c(M+1)k_G(a,b). \tag{3.5}$$

If there are no h'-coarse sequences on the arc, by the convention of coarse length, its h'-coarse length is zero, and the above inequality still holds.

On the other hand, combining the definition of a (M,C)-coarsely quasihyperbolic mapping and the inequality $k_G(a,b)\ge 2C+1$, we obtain

$$k_{G'}\left(f(a),f(b)\right)\ge\frac{k_G(a,b)-C}{M}\ge\frac{k_G(a,b)}{2M}. \tag{3.6}$$

Combining inequalities (3.5) and (3.6), we get

$$\begin{aligned}\ell_{k_{G'}}\left(f\left(\gamma[a,b]\right),h'\right) &\le c\left(M+1\right)k_G\left(a,b\right)\\ &\le 2cM\left(M+1\right)k_{G'}\left(f\left(a\right),f\left(b\right)\right)\\ &= c'k_{G'}\left(f\left(a\right),f\left(b\right)\right).\end{aligned}$$

Synthesizing the above two cases, we can prove that $f\left(\gamma\right)$ is a $\left(c',h'\right)$-solid arc in G', where

$$c'=2cM\left(M+1\right),\quad h'=cM\left(2C+1\right)+C.$$

Since $f^{-1}:G'\to G$ also satisfies the definition of a $\left(M,C\right)$-coarsely quasihyperbolic mapping, repeating the above argument for f^{-1} completes the proof.

The converse to Theorem3.1, based only on two-sided preservation of solid arcs, need not hold.

Example 3.2

Take $X=Y=\mathbb{R}$ and $G=G'=\left(-1,1\right)$. Let $x\in G$, then $\delta_G\left(x\right)=1-\left|x\right|$. Define

$$T\left(x\right)=\operatorname{sgn}\left(x\right)\left[-\log\left(1-\left|x\right|\right)\right].$$

Direct integration gives

$$k_G\left(x,y\right)=\left|T\left(x\right)-T\left(y\right)\right|,$$

So $T:\left(G,k_G\right)\to\left(\mathbb{R},\left|\cdot\right|\right)$ is an isometry. Let

$$H\left(t\right)=t^3,\qquad f=T^{-1}\circ H\circ T.$$

Both $f:G\to G'$ is an increasing homeomorphisms. In G every arc in a real interval is monotone and is a quasihyperbolic geodesic arc. Thus f and f^{-1} send every quasigeodesic arc to a quasigeodesic arc, hence to a $\left(1,0\right)$-solid arc.

However, for $x_n=T^{-1}\left(n\right)$ and $y=0$,

$$k_G\left(x_n,0\right)=\left|T\left(x_n\right)-T\left(0\right)\right|=n$$

and

$$k_G\left(f\left(x_n\right),f\left(0\right)\right)=\left|T\left(f\left(x_n\right)\right)-T\left(f\left(0\right)\right)\right|=n^3.$$

If f were $\left(M,C\right)$-coarsely quasihyperbolic mapping, one would have

$$n^3\le Mn+C$$

for every positive integer n, which is impossible. Therefore, even when the images of quasigeodesic arc is $\left(c,h\right)$-solid arc, the mapping is not $\left(M,C\right)$-coarsely quasihyperbolic mapping. Hence the two-sided arc-preservation property alone does not characterize coarsely quasihyperbolic mappings, even on a one-dimensional uniform domain.

4. A Two-Sided Coarse-Length Criterion for Coarsely Quasihyperbolic Mappings

A theorems by Väisälä [11] gives:

1) Suppose that $f:G\to G'$ is a homeomorphism and that G' is c_0-uni-

form. Suppose also that the image of each 10-neargeodesic in G is (c,h)-solid in G'. Then f is C-coarsely M-Lipschitz in the QH metric with (M,C) depending only on $\upsilon=(c_0,c,h)$.

2) Suppose that $f:G\to G'$ is a homeomorphism and that G and G' are c_0-uniform. In the subsequent corollary, it is concluded that if f is (M,C)-coarsely quasihyperbolic mapping, then The image of each 10-neargeodesic in G is (c,h)-solid in G' and f^{-1} also satisfies the conclusion. Where G and G' are subsets of Banach spaces.

Therefore, this establishes the equivalence between solid arc and coarsely quasihyperbolic mappings in Banach spaces.

While the aforementioned counterexample shows that the converse of Theorem 3.1 fails, it is established in [11] that neargeodesics and (M,C)-coarsely quasihyperbolic are equivalent. This naturally leads us to ponder: combined with the preservation of its image properties, are necessary and sufficient for it to be equivalent to an (M,C)-coarsely quasihyperbolic mapping? Consequently, we obtain an equivalence theorem stating that L-short arcs are equivalent to (M,C)-coarsely quasihyperbolic mappings.

Theorem 4.1

Fix $L>0$. Let $f:G\to G'$ be a homeomorphism, and assume that every pair of points in either G or G' can be joined by a arc of finite quasihyperbolic length. The following conditions are equivalent:

1) f is a (M,C)-coarsely quasihyperbolic mapping for some $M\geq 1, C\geq 0$;

2) There exists constants $A_1\geq 1, A_2\geq 1$ and $h_1\geq 1, h_2\geq 0$, independent of the endpoints and the chosen arc, such that:

① for every $x,y\in G$ and every L-short arc $\gamma\subset G$ joining them,

$$\ell_{k_{G'}}\left(f(\gamma),h_1\right)\leq A_1\left(k_G(x,y)+L\right); \tag{4.1}$$

② for every $u,v\in G'$ and every L-short arc $\beta\subset G'$ joining them,

$$\ell_{k_G}\left(f^{-1}(\beta),h_2\right)\leq A_2\left(k_{G'}(u,v)+L\right). \tag{4.2}$$

Proof: 1) $\Rightarrow$ 2) Let f be a (M,C)-coarsely quasihyperbolic mapping, and take

$$h_1=h_2=\max\{1,ML+C\},\quad A_2=A_2=M+\frac{C}{L}.$$

Let $\gamma\subset G$ be a L-short arc connecting x and y, we have

$$\ell_{k_G}(\gamma)\leq k_G(x,y)+L. \tag{4.3}$$

Set $h=\max\{1,ML+C\}$, and let $(z_0',\cdots,z_n')$ be an arbitrary h-coarse sequence on $f(\gamma)$. Denote $z_i=f^{-1}(z_i')$. For each i, we have

$$ML+C\leq h\leq k_{G'}(z_{i-1}',z_i')\leq Mk_G(z_{i-1},z_i)+C.$$

Thus $k_G(z_{i-1},z_i)\geq L$, and hence

$$\sum_{i=1}^{n}k_G(z_{i-1},z_i)\geq nL. \tag{4.4}$$

Again, by the definition of a coarsely quasihyperbolic mapping and inequalities

(4.3) and (4.4), we obtain

$$\sum_{i=1}^{n} k_{G'}\left(z'_{i-1}, z'_i\right) \le M\sum_{i=1}^{n} k_G\left(z_{i-1}, z_i\right) + nC \le \left(M + \frac{C}{L}\right)\sum_{i=1}^{n} k_G\left(z_{i-1}, z_i\right)$$
$$\le \left(M + \frac{C}{L}\right)\ell_{k_G}(\gamma) \le \left(M + \frac{C}{L}\right)\left(k_G(x,y) + L\right).$$

Taking the supremum yields inequality (4.1). Since the inverse mapping of a (M,C)-coarsely quasihyperbolic mapping is also a (M,C)-coarsely quasihyperbolic mapping, inequality (4.2) follows analogously. Thus, 1) $\Rightarrow$ 2) is proved.

2) $\Rightarrow$ 1). Take arbitrary $x, y \in G$. Let $\gamma \subset G$ be a L-short arc connecting x and y. If $k_{G'}\left(f(x), f(y)\right) \ge h_1$, then the sequence $\left(f(x), f(y)\right)$ is a h_1-coarse sequence on $f(\gamma)$. Therefore, using inequality (4.1), we obtain

$$k_{G'}\left(f(x), f(y)\right) \le \ell_{k_{G'}}\left(f(\gamma), h_1\right) \le A_1\left(k_G(x,y) + L\right).$$

If $k_{G'}\left(f(x), f(y)\right) < h_1$, then obviously

$$k_{G'}\left(f(x), f(y)\right) \le A_1 k_G(x,y) + h_1.$$

Let $B_1 = \max\left\{A_1 L, h_1\right\}$. From the above two cases, we obtain

$$k_{G'}\left(f(x), f(y)\right) \le A_1 k_G(x,y) + B_1. \tag{4.5}$$

Similarly, applying the same argument to f^{-1} and inequality (4.2), and letting $B_2 = \max\left\{A_2 L, h_2\right\}$, we have for any $u, v \in G'$

$$k_G\left(f^{-1}(u), f^{-1}(v)\right) \le A_2 k_{G'}(u,v) + B_2. \tag{4.6}$$

Taking $u = f(x)$ and $v = f(y)$, inequality (4.6) yields

$$k_{G'}\left(f(x), f(y)\right) \ge \frac{k_G(x,y) - B_2}{A_2}. \tag{4.7}$$

Let $M = \max\left\{1, A_1, A_2\right\}$, $C = \max\left\{B_1, B_2\right\}$.

If $k_G(x,y) \le C$, then trivially

$$\frac{k_G(x,y) - C}{M} \le 0 \le k_{G'}\left(f(x), f(y)\right).$$

If $k_G(x,y) > C$, then $k_G(x,y) - C > 0$, and $A_2 \le M$, $B_2 \le C$. Thus,

$$k_{G'}\left(f(x), f(y)\right) \ge \frac{k_G(x,y) - B_2}{A_2} \ge \frac{k_G(x,y) - C}{M}.$$

Combining this with inequalities (4.5) and (4.7), for any $x, y \in G$, we obtain

$$\frac{k_G(x,y) - C}{M} \le k_{G'}\left(f(x), f(y)\right) \le M k_G(x,y) + C.$$

Therefore, f is a (M,C)-coarsely quasihyperbolic mapping.

Hence, Theorem 4.1 is proved.

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Author Contributions

All authors have accepted responsibility for the entire content of this manuscript and consented to its submission to the journal, reviewed all the results, and approved the final version of the manuscript. All authors contributed to the study conception and design.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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