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Treating Fractional Boundary Value Problems via Inverse Fractional Operator and Decomposition Method

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Abstract

This paper aims to apply the standard Adomian decomposition method and its reliable modification to numerically treat the class of two-point boundary value problems (BVPs) and two-point initial-boundary value problems (IBVPs) using fractional partial differential equations (FPDEs). A special fractional inversion operator is used to simplify the decomposition process and further construct an optimal closed-form solution of the governing IBVP after exhausting the imposed two-point Dirichlet boundary conditions. Numerous descriptive testing fractional problems are presented to exhibit the efficacy of the devised approach, all of which lead to exact analytical solutions in most cases or optimal closed-form solutions when exact solutions are not achievable. The obtained solutions demonstrate that the proposed operator is accurate and suitable for solving FPDEs amidst the prescription of suitable initial and boundary conditions. Thus, several comparison tables are reported to assess the effectiveness of the devised method over other open methods via absolute error analysis.

Keywords

Fractional Derivatives, Fractional Inversion Operator, Decomposition Methods, Optimal Closed-Form Solutions, Initial-Boundary Value Problems

1. Introduction

The concept of non-integer calculus, which broadens the principles of integration and differentiation to fractional orders, has captivated mathematicians for over three centuries. Its historical cause is hinted at in the late seventeenth century, when scholars began to explore the possibility of defining a derivative of $\frac{1}{2}$

order. This intriguing question, initially posed by Leibniz in correspondence with L'Hôpital in 1695, initiated a line of inquiry that persisted through the works of many eminent figures, including Leonhard Euler, Liouville, and Riemann. Over time, their collective contributions established the theoretical basis of what is now recognized as fractional calculus [1]. Fractional calculus has emerged as a significant area of research due to its extensive applications across various disciplines, including biomedical engineering, hydrology, probability theory, finance, and electrochemistry. Over recent decades, it has transitioned from a theoretical concept to an effective tool capable of modeling complex dynamical behaviors in different scientific areas, such as engineering, plasma, seismology, aerodynamics, chaotic and fractal dynamics, signal and image processing, artificial intelligence, and control theory. Numerous studies have demonstrated the efficacy of fractional calculus in characterizing intricate dynamical processes. For example, Magin [2] developed fractional-order models for biological tissues, illustrating that fractional-order derivatives can effectively describe the viscoelastic, electrical, and diffusive properties of biological materials. In the financial sector, Balci [3] proposed a fractional integro-differential model to analyze the interactions among financial agents within a stock-market network, uncovering global memory effects and equilibrium-seeking tendencies consistent with empirical market data. Furthermore, fractional calculus has been widely employed in viscoelasticity [4], chaotic systems [5], and various areas of science and engineering [6], highlighting its versatility in describing complex phenomena. Fractional differential equations (FDEs) involve differentiation orders that are not whole numbers. Recently, these equations have become crucial in both applied and theoretical contexts across a range of engineering and scientific professions, such as biology [7], epidemiology [8] [9], and control theory [10]. Moreover, boundary-value problems (BVPs) related to fractional partial differential equations (FPDEs) have, in recent times, captured the interest and guided the research activities of numerous theorists and experimentalists in both the applied and pure sciences. These problems are pivotal in advancing the understanding of systems characterized by nonlocal interactions and persistent memory effects. The consideration of non-integer order derivatives alongside boundary conditions permits the creation of more accurate and adaptable models that can effectively represent hereditary properties, spatial diversity, and atypical transport phenomena, which are often not represented by traditional models. For instance, anomalous diffusion in heterogeneous materials can be accurately represented by space-time fractional diffusion equations defined on bounded domains [11]. Similarly, a fractional Poisson equation in two dimensions has been extensively employed in physical and engineering applications and has been numerically solved using Legendre wavelet approaches upon imposing Dirichlet conditions [12]. Furthermore, space-time fractional advection-diffusion equations have been formulated to model pollutant transport in porous media, where Dirichlet boundaries specify the concentration values at the system edges [13]. These examples underscore the significance and versatility of FPDEs with boundary conditions in modeling realistic processes in diverse fields of engineering and science.

FPDEs with boundary conditions frequently result in intricate mathematical formulations that are challenging or even impossible to solve analytically. As a result, the development of efficient and reliable analytical-numerical techniques becomes imperative. Thus, over the past few decades, numerous robust methods have been proposed to get hold of approximate or exact solutions for FPDEs. In particular, among the methods, the Adomian decomposition method (ADM) [14]-[16] has been effectively employed to solve various fractional BVPs, including non-integer telegraph and diffusion waves. The homotopy perturbation method (HPM) [17] offers another efficient semi-analytical scheme for treating FPDEs in finite domains. Additionally, the new iterative method (NIM) has also been adopted to solve FPDEs featuring Dirichlet boundary conditions [18]. Furthermore, spectral-based approaches employing Chebyshev cardinal functions [19] have been shown to produce highly accurate numerical solutions. In addition, finite difference-based algorithms, including the θ-method [20], have been employed for the computational analysis of fractional Poisson-type equations, while extended cubic B-spline functions [21] have proven to be successful in handling non-integer Klein-Gordon equations. These analytical numerical techniques provide flexible and effective approaches for studying a range of physical, biological, and engineering systems governed by FPDEs with initial and boundary conditions. Indeed, among the many analytical and numerical methods for FPDEs, the current work focuses on the application of the ADM. Since its inception by George Adomian in 1984 [22] [23], the ADM has garnered considerable attention due to its simplicity, flexibility, and notable efficiency in solving various models. Accordingly, the ADM has been refined by several scientists, with the view to enhancing its precision, convergence properties, and computational performance, among others. These ongoing developments have significantly enhanced the method's ability to produce accurate series solutions with fewer iterations compared to its original formulation. Subsequent research has concentrated on refining the mathematical structure and analytical foundation of the ADM. For example, in [24], valuable theoretical insights into the construction and analytic summation of the Adomian series were provided for both differential and integral equations, enriching the understanding of its fundamental properties. The ADM has also been effectively employed in modeling and solving various FPDEs, as documented in [25], underscoring its adaptability in fractional analysis. Furthermore, a hybrid extension referred to as the Laplace decomposition method (LDM) was introduced in [26], which couples the ADM with the Laplace transform to accelerate convergence and simplify computation. More recently, the modified Adomian decomposition method (MADM) for systems of nonlinear FPDEs has been featured in [27], representing a significant advancement that offers enhanced accuracy and computational efficiency. The convergence of the ADM series has been widely studied in the literature [28] [29]. More recent works have examined its convergence for

FPDEs [27] [30], and the method has also been applied to these equations subject to Dirichlet boundary conditions [16] [31].

However, this study specifically aims to expand the utilization of the ADM to the realm of fractional initial-boundary value problems (FIBVP) and fractional boundary value problems (FBVPs) for FPDEs that are subjected to two-point Dirichlet boundary conditions, in addition to the supplementary initial conditions. The concept of introducing an inverse operator suitable for such boundary constraints was originally presented by Lesnic in [32], where a specific operator was constructed to solve the Dirichlet heat conduction problem. This operator was later employed and modified by Abdelhalim Ebaid in [33], where the author successfully incorporated it within the ADM to obtain exact and rapidly convergent solutions. In addition, El-Sayed and Gaber, in [31], developed a fractional inverse operator defined on finite domains to handle fractional derivatives. Thus, the current study utilizes the fractional operator introduced by El-Sayed and Gaber alongside employing the ADM and MADM formulations to solve IBVPs for FPDEs. The importance of this study lies in its focus on the limited research related to fractional inverse operators under Dirichlet boundary conditions. By employing this fractional inverse operator within the ADM, our study provides a more effective method for obtaining accurate and rapidly convergent solutions to FPDEs.

Moreover, the paper is arranged in the following structure: Section 2 gives an overview of the fractional operators. In Section 3, the analytical approach of the proposed scheme is detailed. Section 4 provides a demonstration of the method's efficiency through a series of illustrative examples. Finally, some concluding notes are appended in Section 5.

2. Preludes

This section outlines certain preludes associated with the fractional derivatives and integrals. In particular, certain definitions and properties for the Caputo and Riemann–Liouville (RL) fractional operators will be outlined; see the nice book by Kilbas for more on the theory and application of non-integer calculus [34].

Definition 2.1. The RL fractional integrals $\left(I_{a^+}^{\alpha} y\right)(x)$ when $x > a$ and $\left(I_{b^-}^{\alpha} y\right)(x)$ when $x < b$, of the fractional-order $\alpha\left(\Re(\alpha) > 0\right) \in \mathbb{C}$, are sequentially defined as follows:

$$\left(I_{a^+}^{\alpha} y\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x y(t)(x-t)^{\alpha-1}\,\mathrm{d}t, \tag{1}$$

and

$$\left(I_{b^-}^{\alpha} y\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b y(t)(t-x)^{\alpha-1}\,\mathrm{d}t. \tag{2}$$

Definition 2.2. The RL fractional derivatives $\left(D_{a^+}^{\alpha} y\right)(x)$ when $x > a$ and $\left(D_{b^-}^{\alpha} y\right)(x)$ when $x < b$, of the fractional-order $\alpha\left(\Re(\alpha) \geq 0\right) \in \mathbb{C}$, are sequentially defined (for $n=\left[\Re(\alpha)\right]+1$) as follows:

$$\left(D_{a^+}^{\alpha}y\right)(x):=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\int_a^x y(t)(x-t)^{n-\alpha-1}\mathrm{d}t=\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(I_{a^+}^{n-\alpha}y\right)(x), \quad (3)$$

and

$$\left(D_{b^-}^{\alpha}y\right)(x):=\frac{1}{(n-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\int_x^b y(t)(t-x)^{n-\alpha-1}\mathrm{d}t=\left(-\frac{\mathrm{d}}{\mathrm{d}x}\right)^n\left(I_{b^-}^{n-\alpha}y\right)(x). \quad (4)$$

Definition 2.3. The Caputo fractional derivatives $\left({}^C D_{a^+}^{\alpha}y\right)(x)$ when $x>a$ and $\left({}^C D_{b^-}^{\alpha}y\right)(x)$ when $x<b$, of the fractional-order $\alpha\left(\Re(\alpha)\geq 0\right)\in\mathbb{C}$, are sequentially defined for $\alpha\notin\mathbb{N}_0$ as follows:

$$\left({}^C D_{a^+}^{\alpha}y\right)(x):=\frac{1}{\Gamma(n-\alpha)}\int_a^x y^{(n)}(t)(x-t)^{n-\alpha-1}\mathrm{d}t=\left(I_{a^+}^{n-\alpha}D^n y\right)(x), \quad (5)$$

and

$$\left({}^C D_{b^-}^{\alpha}y\right)(x):=\frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b y^{(n)}(t)(t-x)^{n-\alpha-1}\mathrm{d}t:=(-1)^n\left(I_{b^-}^{n-\alpha}D^n y\right)(x). \quad (6)$$

Additionally, if $\alpha=n\in\mathbb{N}_0$, the latter definitions sequentially reduce to the following:

$$\left({}^C D_{a^+}^{n}y\right)(x)=y^{(n)}(x), \quad (7)$$

$$\left({}^C D_{b^-}^{n}y\right)(x)=(-1)^n y^{(n)}(x). \quad (8)$$

Moreover, when $\alpha=0$, the subsequent fractional derivatives coincide as:

$$\left({}^C D_{a^+}^{0}y\right)(x)=y(x)=\left({}^C D_{b^-}^{0}y\right)(x). \quad (9)$$

Property 2.1. The following are some properties for the Caputo and RL derivatives:

(i) Given a constant $k\in\mathbb{R}$, then $I^{\alpha}k=\dfrac{x^{\alpha}k}{\Gamma(\alpha+1)}$.

(ii) Given a constant $k\in\mathbb{R}$, then ${}^C D_*^{\alpha}k=0$.

(iii) Given the expression $x^n,\alpha>0,n>-1,x>0$, then

$$I^{\alpha}x^n=\frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}x^{n+\alpha}.$$

(iv) Given the expression x^n, then

$${}^C D_*^{\alpha}x^n=\begin{cases}\dfrac{\Gamma(1+n)}{\Gamma(1+n-\alpha)}x^{n-\alpha}, & n>m-1,\\ 0, & n\leq m-1.\end{cases}$$

(v) $\left(I^{\alpha}\,{}^C D_*^{\alpha}y\right)(x):=y(x)-\sum_{k=0}^{m-1}\dfrac{x^k y^{(k)}(0^+)}{k!}$, $m-1<\alpha\leq m$.

(vi) $\left(I^{\alpha}\,{}^C D_*^{\beta}y\right)(x):=I^{\beta-\alpha}y(x)-\sum_{k=0}^{m-1}\dfrac{x^{\beta-\alpha+k}y^{(k)}(0^+)}{\Gamma(1+\beta-\alpha+k)}$, $m-1<\alpha,\beta\leq m$, $\alpha>\beta$.

3. Analysis of the Proposed Method

This section outlines the new method for tackling assorted forms of FPDEs. Indeed, the method is based upon incorporating the inverse fractional operator [31] in ADM recurrence [22], thereby hastening convergence of the standard ADM procedure in formulating and analyzing FPDEs.

Thus, in presenting this new method, one considers the fractional orders α,β for $1<\alpha\le 2$ and $0<\beta\le 2$ in the sense of the Caputo fractional derivative through considering the following general FPDE

$$D_x^{\alpha}u(x,t)+D_t^{\beta}u(x,t)+Ru(x,t)+Nu(x,t)=g(x,t),\ x\in(a,b), \tag{10}$$

that is subjected to the following initial and two-point Dirichlet boundary conditions

for $0<\beta\le 1$

$$u(x,0)=f(x), u(a,t)=f_0(t), u(b,t)=f_1(t), a\le x\le b, t>0, \tag{11}$$

for $1<\beta\le 2$

$$u(x,0)=f(x), u(x,c)=g(x), u(a,t)=f_0(t), u(b,t)=f_1(t), 0\le t\le c. \tag{12}$$

where from (10), (11) and (12) $u(x,t)$ is the unknown solution field, D_x^{α} and D_t^{β} represent the Caputo fractional-order derivatives of order α and β, sequentially, R and N denote the remainder linear and nonlinear terms, sequentially, while $g(x,t)$ is the inhomogeneous function that is prescribed; in addition, the functions $f(x)$, $g(x)$, $f_0(t)$, and $f_1(t)$ are all given nice functions.

Moreover, to apply the ADM, equation (10) is re-expressed as follows

$$D_x^{\alpha}u(x,t)=g(x,t)-D_t^{\beta}u(x,t)-Ru(x,t)-Nu(x,t). \tag{13}$$

Then, we propose an approach in which the inverse fractional operator is applied using only the given boundary conditions. The inverse fractional operator I_x^{α} is defined as follows [31]

$$I_x^{\alpha}(\cdot)=\int_a^x\frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}(\cdot)\,\mathrm{d}s-\frac{x-a}{b-a}\int_a^b\frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}(\cdot)\,\mathrm{d}s. \tag{14}$$

Accordingly, deploying the operator I_x^{α} on (13) and utilizing property 2.1 (vi) results in

$$\begin{aligned}u(x,t)&=u(a,t)+\frac{x-a}{b-a}\left[u(b,t)-u(a,t)\right]\\&\quad+I_x^{\alpha}\left[g(x,t)-D_t^{\beta}u(x,t)-Ru(x,t)-Nu(x,t)\right].\end{aligned} \tag{15}$$

Using the given boundary conditions $u(a,t)=f_0(t)$ and $u(b,t)=f_1(t)$, Equation (15) becomes

$$\begin{aligned}u(x,t)&=f_0(t)+\frac{x-a}{b-a}\left[f_1(t)-f_0(t)\right]\\&\quad+I_x^{\alpha}\left[g(x,t)-D_t^{\beta}u(x,t)-Ru(x,t)-Nu(x,t)\right].\end{aligned} \tag{16}$$

Further, the standard ADM proposes that the solution of the latter equation $u(x,t)$ be expressed as an infinite series of components

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t), \tag{17}$$

while the nonlinear term $Nu(x,t)$ is expressed as an infinite series of Adomian polynomials A_n as

$$Nu(x,t)=\sum_{n=0}^{\infty}A_n, \tag{18}$$

where the polynomials A_n are constructed using [22]

$$A_n=\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\left[N\left(\sum_{k=0}^{\infty}u_k(x,t)\lambda^k\right)\right]\Bigg|_{\lambda=0},\quad n\geq 0. \tag{19}$$

Consequently, plugging the latter series expressions for $u(x,t)$ and $Nu(x,t)$ into (16), one gets

$$\begin{aligned}\sum_{n=0}^{\infty}u_n(x,t)&=f_0(t)+\frac{x-a}{b-a}\left[f_1(t)-f_0(t)\right]\\&\quad+I_x^{\alpha}\left[g(x,t)-D_t^{\beta}\left(\sum_{n=0}^{\infty}u_n(x,t)\right)-R\left(\sum_{n=0}^{\infty}u_n(x,t)\right)-\sum_{n=0}^{\infty}A_n\right].\end{aligned} \tag{20}$$

and leading to the standard ADM's recurrent scheme as follows

$$\begin{cases}u_0(x,t)=h(x,t),\\ u_{n+1}(x,t)=-I_x^{\alpha}\left[D_t^{\beta}u_n(x,t)+Ru_n(x,t)+A_n\right],\ n\geq 0,\end{cases} \tag{21}$$

where

$$h(x,t)=f_0(t)+\frac{x-a}{b-a}\left[f_1(t)-f_0(t)\right]+I_x^{\alpha}\left[g(x,t)\right]. \tag{22}$$

In the same fashion, upon deploying the reliable MADM [35], the recurrent scheme expressed in (21) is recast to take the following relation

$$\begin{cases}u_0(x,t)=h_1(x,t),\\ u_1(x,t)=h_2(x,t)-I_x^{\alpha}\left[D_t^{\beta}u_0(x,t)+Ru_0(x,t)+A_0\right],\\ u_{n+1}(x,t)=-I_x^{\alpha}\left[D_t^{\beta}u_n(x,t)+Ru_n(x,t)+A_n\right],\ n\geq 1,\end{cases} \tag{23}$$

where $h_1(x,t)+h_2(x,t)=h(x,t)$ from (22).

Finally, expected closed-form solution $u(x,t)$ is obtained via either the standard ADM (21) or the MADM (23) schemes by approximating by the truncated series, through (17) as follows

$$u(x,t)=\lim_{k\to\infty}\Phi_k,\quad \Phi_k(x,t)=\sum_{n=0}^{k-1}u_n(x,t). \tag{24}$$

4. Illustrative Examples

This section presents some test problems for the two-point fractional boundary value problems (FBVPs) and FIBVPs of FPDEs to verify the efficiency and accuracy of the devised approach. The results obtained by the new approach will then be compared with exact solutions and other methods to assess its reliability.

Example 4.1. Consider the non-homogeneous space-time FBVP

$$D_x^{\frac{3}{2}}u(x,t)+D_t^{\frac{6}{5}}u(x,t)=\frac{5t^{\frac{4}{5}}}{2\Gamma\left(\frac{4}{5}\right)}+\frac{4\sqrt{x}}{\sqrt{\pi}},\ 0\le x\le 1, t\ge 0, \tag{25}$$

$$u(x,0)=x^2, u(x,1)=x^2+1, u(0,t)=t^2, u(1,t)=1+t^2,$$

that satisfies the exact solution $u(x,t)=x^2+t^2$.

Applying the inverse fractional operator $I_x^{\frac{3}{2}}$ defined in (14) to (25) and substituting the related boundary conditions yields

$$u(x,t)=t^2+x+\frac{4}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{4}\left(x^2-x\right)\right]+\left[\frac{10t^{\frac{4}{5}}}{3\sqrt{\pi}\Gamma\left(\frac{4}{5}\right)}\left(x^{\frac{3}{2}}-x\right)\right]-I_x^{\frac{3}{2}}\left[D_t^{\frac{6}{5}}u(x,t)\right].$$

Now, based on the recurrent scheme in (23) for the MADM, the resulting recursive solution yields

$$\begin{cases} u_0(x,t)=t^2+x+\frac{4}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{4}\left(x^2-x\right)\right], \\ u_1(x,t)=\frac{10t^{\frac{4}{5}}}{3\sqrt{\pi}\Gamma\left(\frac{4}{5}\right)}\left(x^{\frac{3}{2}}-x\right)-I_x^{\frac{3}{2}}\left[D_t^{\frac{6}{5}}u_0(x,t)\right], \\ u_{n+1}(x,t)=-I_x^{\frac{3}{2}}\left[D_t^{\frac{6}{5}}u_n(x,t)\right],\ n\ge 1. \end{cases} \tag{26}$$

Therefore, for the first component $u_0(x,t)$ take the form

$$\begin{aligned} u_0(x,t)&=t^2+x+\frac{4}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{4}\left(x^2-x\right)\right] \\ &=t^2+x+x^2-x \\ &=t^2+x^2. \end{aligned}$$

Consequently, when $n=0$, $u_1(x,t)$ is obtained by using the property 2.1 (iv) as

$$\begin{aligned} u_1(x,t)&=\left[\frac{10t^{\frac{4}{5}}}{3\sqrt{\pi}\Gamma\left(\frac{4}{5}\right)}\left(x^{\frac{3}{2}}-x\right)\right]-I_x^{\frac{3}{2}}\left[D_t^{\frac{6}{5}}u_0(x,t)\right] \\ &=\left[\frac{10t^{\frac{4}{5}}}{3\sqrt{\pi}\Gamma\left(\frac{4}{5}\right)}\left(x^{\frac{3}{2}}-x\right)\right]-I_x^{\frac{3}{2}}\left[\frac{2}{\Gamma\left(\frac{9}{5}\right)}t^{\frac{4}{5}}\right] \\ &=\left[\frac{10t^{\frac{4}{5}}}{3\sqrt{\pi}\Gamma\left(\frac{4}{5}\right)}\left(x^{\frac{3}{2}}-x\right)\right]-\left[\frac{10t^{\frac{4}{5}}}{3\sqrt{\pi}\Gamma\left(\frac{4}{5}\right)}\left(x^{\frac{3}{2}}-x\right)\right]=0. \end{aligned}$$

Thus, for all higher-order terms $u_{n+1}(x,t)=0$, $n\ge 1$. Hence, the obtained series solution is expressed as

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=x^2+t^2,$$

which matches the already stated exact analytical solution for the model. What is more, Table 1 presents a comparison between the normalized Bernstein wavelet technique and the proposed approach for different numbers of iterations, the values of m, where the new method has been observed to reveal an accurate solution.

Table 1. Absolute error comparison for dissimilar values of m for Example 4.1.

(x,t)	Normalized Bernstein wavelet method [36]		Present method
	$m=12$	$m=16$	$m=2$
(0.2, 0.2)	0	0	0
(0.4, 0.4)	0	0	0
(0.6, 0.6)	1.11×10^{-16}	0	0
(0.8, 0.8)	0	0	0

Example 4.2. Consider the FBVP for the Poisson equation as follows

$$D_x^{\frac{4}{3}}u(x,t)+D_t^{\frac{3}{2}}u(x,t)=\frac{3x^{\frac{2}{3}}t(t-2)}{\Gamma\left(\frac{2}{3}\right)}+\frac{4\sqrt{t}x(x-2)}{\sqrt{\pi}},\quad 0\le x,t\le 2, \tag{27}$$
$$u(x,0)=0,\ u(x,2)=0,\ u(0,t)=0,\ u(2,t)=0,$$

that satisfies the exact solution $u(x,t)=xt(t-2)(x-2)$.

Applying the inverse fractional operator $I_x^{\frac{4}{3}}$ defined in (14) to (27) and substituting the boundary conditions yields

$$u(x,t)=\frac{3t(t-2)}{\Gamma\left(\frac{2}{3}\right)}\left[\frac{\Gamma\left(\frac{5}{3}\right)}{2}\left(x^2-2x\right)\right]+\frac{4\sqrt{t}}{\sqrt{\pi}}\left[\frac{2}{\Gamma\left(\frac{13}{3}\right)}\left(x^{\frac{10}{3}}-2^{\frac{7}{3}}x\right)\right.$$
$$\left.-\frac{2}{\Gamma\left(\frac{10}{3}\right)}\left(x^{\frac{7}{3}}-2^{\frac{4}{3}}x\right)\right]-I_x^{\frac{4}{3}}\left[D_t^{\frac{3}{2}}u(x,t)\right],$$

which leads to the consequent ADM recursive relation as follows

$$\begin{cases} u_0(x,t)=\dfrac{3t(t-2)}{\Gamma\left(\frac{2}{3}\right)}\left[\dfrac{\Gamma\left(\frac{5}{3}\right)}{2}\left(x^2-2x\right)\right]=xt(t-2)(x-2), \\ u_1(x,t)=\dfrac{4\sqrt{t}}{\sqrt{\pi}}\left[\dfrac{2}{\Gamma\left(\frac{13}{3}\right)}\left(x^{\frac{10}{3}}-2^{\frac{7}{3}}x\right)-\dfrac{2}{\Gamma\left(\frac{10}{3}\right)}\left(x^{\frac{7}{3}}-2^{\frac{4}{3}}x\right)\right]-I_x^{\frac{4}{3}}\left[D_t^{\frac{3}{2}}u_0(x,t)\right], \\ u_{n+1}(x,t)=-I_x^{\frac{4}{3}}\left[D_t^{\frac{3}{2}}u_n(x,t)\right],\ n\ge 1. \end{cases}$$

Consequently, the component $u_1(x,t)$ is obtained by applying the property 2.1 (iv) as

$$
\begin{aligned}
u_1(x,t) &= \frac{4\sqrt{t}}{\sqrt{\pi}}\left[\frac{2}{\Gamma\left(\frac{13}{3}\right)}\left(x^{\frac{10}{3}}-2^{\frac{7}{3}}x\right)-\frac{2}{\Gamma\left(\frac{10}{3}\right)}\left(x^{\frac{7}{3}}-2^{\frac{4}{3}}x\right)\right]-I_x^{\frac{4}{3}}\left[D_t^{\frac{3}{2}}u_0(x,t)\right] \\
&= \frac{4\sqrt{t}}{\sqrt{\pi}}\left[\frac{2}{\Gamma\left(\frac{13}{3}\right)}\left(x^{\frac{10}{3}}-2^{\frac{7}{3}}x\right)-\frac{2}{\Gamma\left(\frac{10}{3}\right)}\left(x^{\frac{7}{3}}-2^{\frac{4}{3}}x\right)\right]-I_x^{\frac{4}{3}}\left[\frac{4\sqrt{t}}{\sqrt{\pi}}x(x-2)\right] \\
&= \frac{4\sqrt{t}}{\sqrt{\pi}}\left[\frac{2}{\Gamma\left(\frac{13}{3}\right)}\left(x^{\frac{10}{3}}-2^{\frac{7}{3}}x\right)-\frac{2}{\Gamma\left(\frac{10}{3}\right)}\left(x^{\frac{7}{3}}-2^{\frac{4}{3}}x\right)\right] \\
&\quad -\frac{4\sqrt{t}}{\sqrt{\pi}}\left[\frac{2}{\Gamma\left(\frac{13}{3}\right)}\left(x^{\frac{10}{3}}-2^{\frac{7}{3}}x\right)-\frac{2}{\Gamma\left(\frac{10}{3}\right)}\left(x^{\frac{7}{3}}-2^{\frac{4}{3}}x\right)\right] \\
&= 0.
\end{aligned}
$$

Thus, all higher-order components vanish, so $u_{n+1}(x,t)=0$ for $n\geq 1$. Hence, the obtained series components are summed as

$$
u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=xt(t-2)(x-2),
$$

which coincides the previously mentioned exact solution for the model. In contrast, the present method yields the exact analytical solution directly from only two components, while in [37] certain numerical methods approached the exact solution only as m increased for $m=16,32,64$.

Example 4.3. Consider the FIBVP for the heat equation as follows

$$
\begin{aligned}
&D_t u(x,t)=\frac{\Gamma(3-\alpha)x^{\alpha}}{2}D_x^{\alpha}u(x,t)-x^2\left(\sin(t)+\cos(t)\right),\ 0<x<1, 0<t, \\
&u(x,0)=x^2,\ u(0,t)=0,\ u(1,t)=\cos(t),
\end{aligned}
\tag{28}
$$

where $\alpha\in(1,2]$, and admits an exact solution as $u(x,t)=x^2\cos(t)$.

Applying the inverse fractional operator I_x^{α} defined in (14) to (28) and substituting the imposed conditions reveals

$$
u(x,t)=x\cos(t)+\left[\left(x^2-x\right)\left(\sin(t)+\cos(t)\right)\right]+I_x^{\alpha}\left[\frac{2}{\Gamma(3-\alpha)x^{\alpha}}D_t u(x,t)\right].
$$

Now, based on the recursive relation in (23), the MADM yields the consequent iterative scheme as

$$
\begin{cases}
u_0(x,t)=x\cos(t)+\left(x^2-x\right)\cos(t)=x^2\cos(t), \\
u_1(x,t)=\left(x^2-x\right)\sin(t)+I_x^{\alpha}\left[\dfrac{2}{\Gamma(3-\alpha)x^{\alpha}}D_t u_0(x,t)\right], \\
u_{n+1}(x,t)=I_x^{\alpha}\left[\dfrac{2}{\Gamma(3-\alpha)x^{\alpha}}D_t u_n(x,t)\right],\ n\geq 1.
\end{cases}
\tag{29}
$$

Accordingly, $u_1(x,t)$ is obtained from the latter scheme as

$$\begin{aligned} u_1(x,t) &= \left(x^2-x\right)\sin(t) + I_x^\alpha\left[\frac{2}{\Gamma(3-\alpha)x^\alpha}D_t u_0(x,t)\right] \\ &= \left(x^2-x\right)\sin(t) + I_x^\alpha\left[\frac{-2x^{2-\alpha}}{\Gamma(3-\alpha)}\sin(t)\right] \\ &= \left(x^2-x\right)\sin(t) - \left(x^2-x\right)\sin(t) = 0. \end{aligned}$$

Thus, for all higher-order terms $u_{n+1}(x,t)=0$ for $n \ge 1$.

Hence, the recurrent solution sums up to

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = x^2\cos(t),$$

which coincides with the reported exact analytical solution of (28). Moreover, the exactness of the devised approach depends only on the boundary conditions as demonstrated through comparing the absolute errors of the new method with those of the hybrid B-spline collocation method [38] and the reproducing kernel method [39] at $t = 0.1, 0.5$, and 1.0 for dissimilar fixed values of α, as reported in **Table 2**.

Table 2. Absolute error comparison for the three methods for Example 4.3.

α	t	x	Hybrid B-spline collocation method [38] ($m = 50$)	Reproducing kernel method [39] ($m = 50$)	Present method ($m = 2$)
1.2	0.1	0.2	4.214500×10^{-6}	8.19791×10^{-7}	0
		0.4	1.683760×10^{-5}	9.79366×10^{-7}	0
		0.6	3.786940×10^{-5}	7.15178×10^{-7}	0
		0.8	6.729740×10^{-5}	3.11357×10^{-7}	0
		1.0	9.311570×10^{-5}	0	0
	0.5	0.2	1.264265×10^{-4}	2.66663×10^{-6}	0
		0.4	5.044576×10^{-4}	3.58987×10^{-6}	0
		0.6	1.125615×10^{-3}	3.25967×10^{-6}	0
		0.8	1.797010×10^{-3}	2.09928×10^{-6}	0
		1.0	2.060472×10^{-3}	0	0
	1.0	0.2	6.186622×10^{-4}	3.99230×10^{-6}	0
		0.4	2.424003×10^{-3}	5.75924×10^{-6}	0
		0.6	4.742281×10^{-3}	6.25942×10^{-6}	0
		0.8	6.271795×10^{-3}	5.01380×10^{-6}	0
		1.0	6.707507×10^{-3}	0	0
1.4	0.1	0.2	4.212000×10^{-6}	1.14526×10^{-6}	0
		0.4	1.683260×10^{-5}	1.39628×10^{-6}	0
		0.6	3.786180×10^{-5}	1.04843×10^{-6}	0
		0.8	6.703560×10^{-5}	4.65691×10^{-7}	0
		1.0	8.856190×10^{-5}	0	0

Continued

	0.5	0.2	1.262725×10^{-4}	2.44965×10^{-6}	0
		0.4	5.030643×10^{-4}	3.03668×10^{-6}	0
		0.6	1.081010×10^{-3}	2.45192×10^{-6}	0
		0.8	1.614415×10^{-3}	1.38966×10^{-6}	0
		1.0	1.817044×10^{-3}	0	0
	1.0	0.2	6.147080×10^{-4}	2.67096×10^{-6}	0
		0.4	2.256320×10^{-3}	3.47873×10^{-6}	0
		0.6	4.044128×10^{-3}	3.71983×10^{-6}	0
		0.8	5.201407×10^{-3}	2.99390×10^{-6}	0
		1.0	5.555100×10^{-3}	0	0
1.6	0.1	0.2	4.209500×10^{-6}	1.98759×10^{-6}	0
		0.4	1.682760×10^{-5}	2.56234×10^{-6}	0
		0.6	3.784650×10^{-5}	2.08248×10^{-6}	0
		0.8	6.588950×10^{-5}	1.00788×10^{-6}	0
		1.0	8.337810×10^{-5}	0	0
	0.5	0.2	1.260839×10^{-4}	1.71001×10^{-6}	0
		0.4	4.941251×10^{-4}	2.03502×10^{-6}	0
		0.6	1.002314×10^{-3}	2.08562×10^{-6}	0
		0.8	1.430260×10^{-3}	1.62926×10^{-6}	0
		1.0	1.590903×10^{-3}	0	0
	1.0	0.2	5.968458×10^{-4}	3.74175×10^{-6}	0
		0.4	2.000868×10^{-3}	6.47819×10^{-6}	0
		0.6	3.397228×10^{-3}	6.72774×10^{-6}	0
		0.8	4.305544×10^{-3}	4.38889×10^{-6}	0
		1.0	4.595036×10^{-3}	0	0

Example 4.4. Consider the FIBVP for the heat equation

$$D_t u(x,t)=\Gamma\left(\frac{3}{2}\right)x^{\frac{1}{2}}D_x^{\frac{3}{2}}u(x,t)+\left(x^2+1\right)\cos(1+t)-2x\sin(1+t),0<x<1,$$
$$0<t,?\ u(x,0)=\left(x^2+1\right)\sin(1),\ u(0,t)=\sin(1+t),\ u(1,t)=2\sin(1+t), \tag{30}$$

which admits the exact solution $u(x,t)=\left(x^2+1\right)\sin(1+t)$.

Applying the inverse fractional operator $I_x^{\frac{3}{2}}$ defined in (14) to (30) and substituting the imposed conditions yields

$$u(x,t)=(x+1)\sin(1+t)-\left[\frac{1}{4}\left(x^3-x\right)\cos(1+t)-\left(x^2-x\right)\sin(1+t)\right]$$
$$+I_x^{\frac{3}{2}}\left[\frac{1}{\Gamma\left(\frac{3}{2}\right)x^{\frac{1}{2}}}D_t u(x,t)\right]$$

Next, based on the MADM recursive relation in (23), one obtains the recurrent solution scheme as

$$\begin{cases} u_0(x,t) = (x+1)\sin(1+t) + (x^2 - x)\sin(1+t) = (x^2+1)\sin(1+t), \\ u_1(x,t) = -\frac{1}{4}(x^3 - x)\cos(1+t) + I_x^{\frac{3}{2}}\left[\frac{1}{\Gamma\left(\frac{3}{2}\right)x^{\frac{1}{2}}} D_t u_0(x,t)\right], \\ u_{n+1}(x,t) = I_x^{\frac{3}{2}}\left[\frac{1}{\Gamma\left(\frac{3}{2}\right)x^{\frac{1}{2}}} D_t u_n(x,t)\right], \ n \geq 1. \end{cases}$$

Accordingly, the component $u_1(x,t)$ is obtained from the above recursive relation as follows

$$\begin{aligned} u_1(x,t) &= I_x^{\frac{3}{2}}\left[\frac{1}{\Gamma\left(\frac{3}{2}\right)x^{\frac{1}{2}}} D_t u_0(x,t)\right] - \left[\frac{1}{4}(x^3 - x)\cos(1+t)\right] \\ &= I_x^{\frac{3}{2}}\left[\frac{(x^2+1)\cos(1+t)}{\Gamma\left(\frac{3}{2}\right)x^{\frac{1}{2}}}\right] - \left[\frac{1}{4}(x^3 - x)\cos(1+t)\right] \\ &= \left[\frac{1}{4}(x^3 - x)\cos(1+t)\right] - \left[\frac{1}{4}(x^3 - x)\cos(1+t)\right] = 0. \end{aligned}$$

Thus, all higher-order components vanish, so $u_{n+1}(x,t) = 0$ for $n \geq 1$. Hence, upon summing up the above iterates, one gets

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = (x^2+1)\sin(1+t).$$

which agrees with the reported exact analytical solution for the FIBVP. Consequently, **Table 3** presents a comparison of results, through absolute errors, between the devised new approach and others in the existing literature; specifically, the shifted Legendre tau method [40] and the Chebyshev finite difference method [41]. Notably, the proposed method achieves an exact solution with only two terms, in contrast to competing methods, thereby confirming its higher accuracy and computational efficiency.

Table 3. Absolute error comparison of the methods for Example 0.4 when $t = 1$.

x	**Shifted Legendre tau method [40]**	**Chebyshev finite difference method [41]**	**Present method**
	$m = 7$	$m = 7$	$m = 2$
0.1	4.66×10^{-5}	1.86×10^{-8}	0
0.2	7.74×10^{-5}	1.23×10^{-8}	0
0.3	5.00×10^{-5}	6.94×10^{-9}	0

Continued

0.4	2.30×10^{-5}	1.26×10^{-8}	0
0.5	2.74×10^{-5}	1.86×10^{-8}	0
0.6	4.38×10^{-5}	1.24×10^{-8}	0
0.7	3.87×10^{-5}	6.29×10^{-10}	0
0.8	1.01×10^{-5}	1.01×10^{-8}	0
0.9	3.35×10^{-6}	4.82×10^{-8}	0

Example 4.5. Consider the FIBVP

$$\begin{gathered} kD_x^{2\beta}u(x,t) = D_t^{\alpha}u(x,t),\ 0<x<l,\ 0<t,\ \alpha\in(0,1],\ \beta\in\left(\frac{1}{2},1\right], \\ u(x,0)=\frac{2x^{2\beta}}{\Gamma(2\beta+1)}, u(0,t)=\frac{2kt^{\alpha}}{\Gamma(\alpha+1)}, u(l,t)=\frac{2l^{2\beta}}{\Gamma(2\beta+1)}+\frac{2kt^{\alpha}}{\Gamma(\alpha+1)}, \end{gathered} \tag{31}$$

that admits the exact solution $u(x,t)=\frac{2x^{2\beta}}{\Gamma(2\beta+1)}+\frac{2kt^{\alpha}}{\Gamma(\alpha+1)}$.

Accordingly, upon applying the inverse fractional operator $I_x^{2\beta}$ defined in (14) to (31) and substituting the prescribed conditions yields the following

$$u(x,t)=\frac{2kt^{\alpha}}{\Gamma(\alpha+1)}+\frac{x}{l}\frac{2l^{2\beta}}{\Gamma(2\beta+1)}+I_x^{2\beta}\left[\frac{1}{k}D_t^{\alpha}u(x,t)\right].$$

Further, the solution of the above equation is recursively obtained by applying the ADM procedure (21) as follows

$$\begin{cases} u_0(x,t)=\dfrac{2kt^{\alpha}}{\Gamma(\alpha+1)}+\dfrac{x}{l}\dfrac{2l^{2\beta}}{\Gamma(2\beta+1)}, \\ u_{n+1}(x,t)=I_x^{2\beta}\left[\dfrac{1}{k}D_t^{\alpha}u_n(x,t)\right],\ n\ge 0. \end{cases}$$

Consequently, we obtain $u_1(x,t)$ from the latter scheme by using the property 2.1 (iv) as

$$\begin{aligned} u_1(x,t)=I_x^{2\beta}\left[\frac{1}{k}D_t^{\alpha}u_0(x,t)\right]=I_x^{2\beta}\left[\frac{1}{k}(2k)\right]=I_x^{2\beta}[2] \\ =\frac{2x^{2\beta}}{\Gamma(2\beta+1)}-\frac{x}{l}\frac{2l^{2\beta}}{\Gamma(2\beta+1)}, \end{aligned}$$

and all other components vanish, that is, $u_n(x,t)=0$ for $n\ge 2$. As a result, the resultant solution is obtained by summing the solution components as

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)=\frac{2x^{2\beta}}{\Gamma(2\beta+1)}+\frac{2kt^{\alpha}}{\Gamma(\alpha+1)},$$

which perfectly aligns with the exact solution of FIBVP. Moreover, the same model was examined in the literature using methods like the homotopy method [17] among others. In particular, El-sayed and Gaber [31] equally got hold of the same exact solution by applying both initial and boundary conditions, whereas authors in [15] proposed an algorithm based only on the initial conditions. Thus, the present method reveals the exact solution of the model using only the boundary conditions, with fewer components and simpler calculations. Notably, upon disregarding the fractional orders, that is, by considering the integer orders $\alpha = 1$ and $\beta = 1$, the acquired fractional expression in the latter solution reduces to the corresponding integer-order heat equation solution as $u(x,t) = x^2 + 2kt$.

Example 4.6. Consider the spatio-temporal FIBVP for the heat equation as follows

$$\begin{aligned}
&D_x^{2\alpha}u(x,t) = D_t^{2\beta}u(x,t),\ 0 < x < 1,\ 0 < t,\ \frac{1}{2} < \alpha, \beta \le 1, \\
&u(x,0) = \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)}, u(0,t) = \frac{2t^{2\beta}}{\Gamma(2\beta+1)}, u(1,t) = \frac{2}{\Gamma(2\alpha+1)} + \frac{2t^{2\beta}}{\Gamma(2\beta+1)},
\end{aligned} \tag{32}$$

that satisfies the exact solution $u(x,t) = \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{2t^{2\beta}}{\Gamma(2\beta+1)}$.

In the same way, with the application of the inverse fractional operator $I_x^{2\alpha}$ (14) on (32), and thereafter substituting the boundary conditions gives

$$u(x,t) = \frac{2t^{2\beta}}{\Gamma(2\beta+1)} + \frac{2x}{\Gamma(2\alpha+1)} + I_x^{2\alpha}\left[D_t^{2\beta}u(x,t)\right],$$

where the latter equation results in the acquisition of the overall ADM recursive scheme as follows

$$\begin{cases} u_0(x,t) = \dfrac{2t^{2\beta}}{\Gamma(2\beta+1)} + \dfrac{2x}{\Gamma(2\alpha+1)}, \\ u_{n+1}(x,t) = I_x^{2\alpha}\left[D_t^{2\beta}u_n(x,t)\right],\ n \ge 0. \end{cases}$$

Consequently, $u_1(x,t)$ is obtained using the property 2.1 (iv) as

$$\begin{aligned} u_1(x,t) &= I_x^{2\alpha}\left[D_t^{2\beta}u_0(x,t)\right] = I_x^{2\alpha}[2] \\ &= \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{2x}{\Gamma(2\alpha+1)}, \end{aligned}$$

while all other higher-order components vanish, that is, $u_n(x,t) = 0$ for $n \ge 2$. This way, the reported resulting actual solution is recovered as [15]

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \frac{2t^{2\beta}}{\Gamma(2\beta+1)} + \frac{2x^{2\alpha}}{\Gamma(2\alpha+1)}.$$

Equally, one notes that upon considering the integer-order $\alpha = \beta = 1$, the latter non-integer solution recasts to the solution of the classical model as $u(x,t) = x^2 + t^2$.

Note. In [15], the same example was solved using the MADM with respect to the time differential operator, where a power series expansion in x was employed to determine the coefficients $a_0, a_1, a_2, \cdots$ before obtaining the final solution. In contrast, the inverse operator $I_x^{2\alpha}$ used in this study yields the exact solution in a much simpler way with fewer computational steps using only the boundary conditions. The vanishing of all higher-order components confirms the effectiveness of the inverse operator $I_x^{2\alpha}$ in simplifying the solution steps and achieving fast convergence.

Example 4.7. Consider the system of FIBVP when $0<\alpha\le 1$, $1<\beta\le 2$ as follows

$$\begin{cases} D_t^{\alpha}u(x,t)-D_x^{\beta}u(x,t)+v(x,t)=t^4-x^3+\dfrac{2t^{2-\alpha}x^4}{\Gamma(3-\alpha)}-\dfrac{24t^2x^{4-\beta}}{\Gamma(5-\beta)}, \\ D_t^{\alpha}v(x,t)-D_x^{\beta}v(x,t)+u(x,t)=t^2x^4+\dfrac{6x^{3-\beta}}{\Gamma(4-\beta)}+\dfrac{24t^{4-\alpha}}{\Gamma(5-\alpha)}, \end{cases} \tag{33}$$

$$x\in(0,1),\ 0<t,\ u(x,0)=0,\ v(x,0)=-x^3,\ u(0,t)=0,\ v(0,t)=t^4, u(1,t)=t^2,\ v(1,t)=t^4-1,$$

which satisfies the exact solution $u(x,t)=t^2x^4$, $v(x,t)=t^4-x^3$.

In the same manner, the application of the fractional inversion operator I_x^{β} (14) on the coupled system (33), in addition to the substitution of the boundary conditions, reveals

$$\begin{cases} u(x,t)=xt^2-\left[\dfrac{t^4(x^{\beta}-x)}{\Gamma(\beta+1)}-\dfrac{6(x^{\beta+3}-x)}{\Gamma(\beta+4)}+\dfrac{48t^{2-\alpha}(x^{\beta+4}-x)}{\Gamma(3-\alpha)\Gamma(\beta+5)}-t^2(x^4-x)\right]+I_x^{\beta}\left[D_t^{\alpha}u(x,t)+v(x,t)\right], \\ v(x,t)=t^4-x-\left[\dfrac{24t^2(x^{\beta+4}-x)}{\Gamma(\beta+5)}+(x^3-x)+\dfrac{24t^{4-\alpha}(x^{\beta}-x)}{\Gamma(5-\alpha)\Gamma(\beta+1)}\right]+I_x^{\beta}\left[D_t^{\alpha}v(x,t)+u(x,t)\right]. \end{cases} \tag{34}$$

Accordingly, the MADM in (23) reveals the resulting scheme for obtaining $u(x,t)$ as

$$\begin{cases} u_0(x,t)=xt^2+t^2(x^4-x)=x^4t^2, \\ u_1(x,t)=-\dfrac{t^4(x^{\beta}-x)}{\Gamma(\beta+1)}+\dfrac{6(x^{\beta+3}-x)}{\Gamma(\beta+4)}-\dfrac{48t^{2-\alpha}(x^{\beta+4}-x)}{\Gamma(3-\alpha)\Gamma(\beta+5)}+I_x^{\beta}\left[D_t^{\alpha}u_0(x,t)+v_0(x,t)\right], \\ u_{n+1}(x,t)=I_x^{\beta}\left[D_t^{\alpha}u_n(x,t)+v_n(x,t)\right],\ n\ge 1, \end{cases}$$

while that of obtaining $v_n(x,t)$ reads

$$\begin{cases} v_0(x,t)=t^4-x-(x^3-x)=t^4-x^3, \\ v_1(x,t)=-\dfrac{24t^2(x^{\beta+4}-x)}{\Gamma(\beta+5)}-\dfrac{24t^{4-\alpha}(x^{\beta}-x)}{\Gamma(5-\alpha)\Gamma(\beta+1)}+I_x^{\beta}\left[D_t^{\alpha}v_0(x,t)+u_0(x,t)\right], \\ v_{n+1}(x,t)=I_x^{\beta}\left[D_t^{\alpha}v_n(x,t)+u_n(x,t)\right],\ n\ge 1. \end{cases}$$

Next, on using the property 2.1 (iv), one computes the components $u_1(x,t)$ and $v_1(x,t)$ as

$$\begin{aligned} u_1(x,t)&=-\frac{t^4(x^{\beta}-x)}{\Gamma(\beta+1)}+\frac{6(x^{\beta+3}-x)}{\Gamma(\beta+4)}-\frac{48t^{2-\alpha}(x^{\beta+4}-x)}{\Gamma(3-\alpha)\Gamma(\beta+5)}+I_x^{\beta}\left[D_t^{\alpha}u_0(x,t)+v_0(x,t)\right] \\ &=-\frac{t^4(x^{\beta}-x)}{\Gamma(\beta+1)}+\frac{6(x^{\beta+3}-x)}{\Gamma(\beta+4)}-\frac{48t^{2-\alpha}(x^{\beta+4}-x)}{\Gamma(3-\alpha)\Gamma(\beta+5)}+I_x^{\beta}\left[\frac{2x^4t^{2-\alpha}}{\Gamma(3-\alpha)}+t^4-x^3\right] \\ &=-\frac{t^4(x^{\beta}-x)}{\Gamma(\beta+1)}+\frac{6(x^{\beta+3}-x)}{\Gamma(\beta+4)}-\frac{48t^{2-\alpha}(x^{\beta+4}-x)}{\Gamma(3-\alpha)\Gamma(\beta+5)}+\left[\frac{48t^{2-\alpha}(x^{\beta+4}-x)}{\Gamma(3-\alpha)\Gamma(\beta+5)}+\frac{t^4(x^{\beta}-x)}{\Gamma(\beta+1)}-\frac{6(x^{\beta+3}-x)}{\Gamma(\beta+4)}\right] \\ &=0, \end{aligned}$$

and

$$
\begin{aligned}
v_1(x,t) &= -\frac{24t^2\left(x^{\beta+4}-x\right)}{\Gamma(\beta+5)} - \frac{24t^{4-\alpha}\left(x^{\beta}-x\right)}{\Gamma(5-\alpha)\Gamma(\beta+1)} + I_x^{\beta}\left[D_t^{\alpha} v_0(x,t) + u_0(x,t)\right] \\
&= -\frac{24t^2\left(x^{\beta+4}-x\right)}{\Gamma(\beta+5)} - \frac{24t^{4-\alpha}\left(x^{\beta}-x\right)}{\Gamma(5-\alpha)\Gamma(\beta+1)} + I_x^{\beta}\left[\frac{24t^{4-\alpha}}{\Gamma(5-\alpha)} + x^4t^2\right] \\
&= -\frac{24t^2\left(x^{\beta+4}-x\right)}{\Gamma(\beta+5)} - \frac{24t^{4-\alpha}\left(x^{\beta}-x\right)}{\Gamma(5-\alpha)\Gamma(\beta+1)} + \left[\frac{24t^{4-\alpha}\left(x^{\beta}-x\right)}{\Gamma(5-\alpha)\Gamma(\beta+1)} + \frac{24t^2\left(x^{\beta+4}-x\right)}{\Gamma(\beta+5)}\right] \\
&= 0.
\end{aligned}
$$

Respectively, while all other components vanish, that is, $u_{n+1}(x,t) = v_{n+1}(x,t) = 0$ for all $n \ge 1$. Whence, upon summing the respective iterates up yields

$$
u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = x^4t^2, \quad v(x,t) = \sum_{n=0}^{\infty} v_n(x,t) = t^4 - x^3.
$$

which aligns with the already stated exact analytical solution of (33). Moreover, it is worth noting that the new approach yields an exact solution of the governing model in just two steps. In contrast, Alfalqi *et al.* [42] recently provided an approximate solution for the same model using advanced neural network frameworks.

Example 4.8. Consider the system of FIBVP when $0<\alpha<1$, $1<\beta<2$ as follows

$$
\begin{cases}
D_t^{\alpha} u(x,t) - D_x^{\beta} u(x,t) + v(x,t) = x + t^2 + \dfrac{t^{1-\alpha}x^2}{\Gamma(2-\alpha)} - \dfrac{2tx^{2-\beta}}{\Gamma(3-\beta)}, \\
D_t^{\alpha} v(x,t) - D_x^{\beta} v(x,t) + u(x,t) = tx^2 + \dfrac{2t^{2-\alpha}}{\Gamma(3-\alpha)}.
\end{cases}
$$

$$
\begin{aligned}
&x \in (0,1),\ 0<t,\ u(x,0)=0,\ v(x,0)=x,\ u(0,t)=0,\ u(0,t)=t^2, \\
&u(1,t)=t,\ u(1,t)=t^2+1,
\end{aligned} \tag{35}
$$

that satisfies the actual solution $u(x,t) = tx^2$, $v(x,t) = t^2 + x$.

Similarly, applying the inverse fractional operator I_x^{β} (14) to the coupled system (35) and substituting the boundary conditions, one gets

$$
\begin{cases}
u(x,t) = xt - \left[\dfrac{\left(x^{\beta+1}-x\right)}{\Gamma(\beta+2)} + \dfrac{t^2\left(x^{\beta}-x\right)}{\Gamma(\beta+1)} + \dfrac{2t^{1-\alpha}\left(x^{\beta+2}-x\right)}{\Gamma(2-\alpha)\Gamma(\beta+3)} - t\left(x^2-x\right)\right] \\
\qquad + I_x^{\beta}\left[D_t^{\alpha} u(x,t) + v(x,t)\right], \\
v(x,t) = t^2 + x - \left[\dfrac{2t\left(x^{\beta+2}-x\right)}{\Gamma(\beta+3)} + \dfrac{2t^{2-\alpha}\left(x^{\beta}-x\right)}{\Gamma(3-\alpha)\Gamma(\beta+1)}\right] \\
\qquad + I_x^{\beta}\left[D_t^{\alpha} v(x,t) + u(x,t)\right].
\end{cases} \tag{36}
$$

In the same way, the MADM algorithm (23) reveals the respective solution schemes for $u(x,t)$ and $v(x,t)$ as follows

$$\begin{cases} u_0(x,t) = xt + t(x^2 - x) = x^2 t, \\ u_1(x,t) = -\dfrac{(x^{\beta+1} - x)}{\Gamma(\beta+2)} - \dfrac{t^2(x^{\beta} - x)}{\Gamma(\beta+1)} - \dfrac{2t^{1-\alpha}(x^{\beta+2} - x)}{\Gamma(2-\alpha)\Gamma(\beta+3)} \\ \qquad\qquad + I_x^{\beta}\left[D_t^{\alpha} u_0(x,t) + v_0(x,t)\right], \\ u_{n+1}(x,t) = I_x^{\beta}\left[D_t^{\alpha} u_n(x,t) + v_n(x,t)\right], \ n \ge 1, \end{cases}$$

and

$$\begin{cases} v_0(x,t) = t^2 + x, \\ v_1(x,t) = -\dfrac{2t(x^{\beta+2} - x)}{\Gamma(\beta+3)} - \dfrac{2t^{2-\alpha}(x^{\beta} - x)}{\Gamma(3-\alpha)\Gamma(\beta+1)} + I_x^{\beta}\left[D_t^{\alpha} v_0(x,t) + u_0(x,t)\right], \\ v_{n+1}(x,t) = I_x^{\beta}\left[D_t^{\alpha} v_n(x,t) + u_n(x,t)\right], \ n \ge 1. \end{cases}$$

Accordingly, upon using the property 2.1 (iv), one computes $u_1(x,t)$ and $v_1(x,t)$ as follows

$$\begin{aligned} u_1(x,t) &= -\frac{(x^{\beta+1} - x)}{\Gamma(\beta+2)} - \frac{t^2(x^{\beta} - x)}{\Gamma(\beta+1)} - \frac{2t^{1-\alpha}(x^{\beta+2} - x)}{\Gamma(2-\alpha)\Gamma(\beta+3)} \\ &\quad + I_x^{\beta}\left[D_t^{\alpha} u_0(x,t) + v_0(x,t)\right] \\ &= -\frac{(x^{\beta+1} - x)}{\Gamma(\beta+2)} - \frac{t^2(x^{\beta} - x)}{\Gamma(\beta+1)} - \frac{2t^{1-\alpha}(x^{\beta+2} - x)}{\Gamma(2-\alpha)\Gamma(\beta+3)} \\ &\quad + I_x^{\beta}\left[\frac{x^2 t^{1-\alpha}}{\Gamma(2-\alpha)} + t^2 + x\right] \\ &= -\frac{(x^{\beta+1} - x)}{\Gamma(\beta+2)} - \frac{t^2(x^{\beta} - x)}{\Gamma(\beta+1)} - \frac{2t^{1-\alpha}(x^{\beta+2} - x)}{\Gamma(2-\alpha)\Gamma(\beta+3)} \\ &\quad + \left[\frac{2t^{1-\alpha}(x^{\beta+2} - x)}{\Gamma(2-\alpha)\Gamma(\beta+3)} + \frac{t^2(x^{\beta} - x)}{\Gamma(\beta+1)} + \frac{(x^{\beta+1} - x)}{\Gamma(\beta+2)}\right] \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} v_1(x,t) &= -\frac{2t(x^{\beta+2} - x)}{\Gamma(\beta+3)} - \frac{2t^{2-\alpha}(x^{\beta} - x)}{\Gamma(3-\alpha)\Gamma(\beta+1)} + I_x^{\beta}\left[D_t^{\alpha} v_0(x,t) + u_0(x,t)\right] \\ &= -\frac{2t(x^{\beta+2} - x)}{\Gamma(\beta+3)} - \frac{2t^{2-\alpha}(x^{\beta} - x)}{\Gamma(3-\alpha)\Gamma(\beta+1)} + I_x^{\beta}\left[\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + x^2 t\right] \\ &= -\frac{2t(x^{\beta+2} - x)}{\Gamma(\beta+3)} - \frac{2t^{2-\alpha}(x^{\beta} - x)}{\Gamma(3-\alpha)\Gamma(\beta+1)} \\ &\quad + \left[\frac{2t^{2-\alpha}(x^{\beta} - x)}{\Gamma(3-\alpha)\Gamma(\beta+1)} + \frac{2t(x^{\beta+2} - x)}{\Gamma(\beta+3)}\right] \\ &= 0. \end{aligned}$$

while all other terms vanish for all $n \ge 1$, which equally further affirms the reported actual solution. In contrast, an approximate solution of the model was reported by Alfalqi *et al.* [42] via the application of the contemporary advanced neural networking algorithm.

5. Conclusion

This study presents a series of FPDEs involving initial and boundary conditions, solved using the standard ADM and its reliable modification with a special inverse operator, which uses only the boundary conditions. The proposed fractional inverse operator plays a central role in simplifying the decomposition process and accelerating convergence in both approaches. All illustrative examples produced exact analytical solutions, affirming the validity, exactness, and efficiency of the devised method. In certain examples, the obtained results matched those reported in the literature, whereas in others, the present method achieved exact solutions, while the referenced numerical approaches provided only approximate results. Notably, in both the standard and reliable versions of the proposed ADM-cased methods, the exact analytical solutions were attained from the first two components, u_0 and u_1, indicating that the devised inverse operator effectively minimizes the computational effort and eliminates the need for higher-order iterations. Hence, the proposed method can be considered a robust and precise analytical tool for tackling a wide range of FPDEs by using only the boundary conditions. Future studies should focus on extending this approach to nonlinear and coupled FPDE systems, reinforcing its applicability in assorted engineering and scientific fields.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Lazarević, M.P., Rapaić, M.R., Šekara, T.B., Mladenov, V. and Mastorakis, N. (2014) Introduction to Fractional Calculus with Brief Historical Background. In: *Advanced Topics on Applications of Fractional Calculus on Control Problems, System Stability and Modeling*, World Scientific and Engineering Academy and Society, 3-16.

[2] Magin, R.L. (2012) Fractional Calculus in Bioengineering: A Tool to Model Complex Dynamics. *Proceedings of the 13th International Carpathian Control Conference (ICCC)*, High Tatras, 28-31 May 2012, 464-469. https://doi.org/10.1109/carpathiancc.2012.6228688

[3] Balcı, M.A. (2020) Fractional Interaction of Financial Agents in a Stock Market Network. *Applied Mathematics and Nonlinear Sciences*, **5**, 317-336. https://doi.org/10.2478/amns.2020.1.00030

[4] Mainardi, F. (2012) An Historical Perspective on Fractional Calculus in Linear Viscoelasticity. *Fractional Calculus and Applied Analysis*, **15**, 712-717. https://doi.org/10.2478/s13540-012-0048-6

[5] Hammouch, Z., Yavuz, M. and Özdemir, N. (2021) Numerical Solutions and Synchronization of a Variable-Order Fractional Chaotic System. *Mathematical Modelling and Numerical Simulation with Applications*, **1**, 11-23. https://doi.org/10.53391/mmnsa.2021.01.002

[6] Debnath, L. (2003) Recent Applications of Fractional Calculus to Science and Engineering. *International Journal of Mathematics and Mathematical Sciences*, **2003**, 3413-3442. https://doi.org/10.1155/s0161171203301486

[7] Hattaf, K. and Yousfi, N. (2020) Global Stability for Fractional Diffusion Equations in Biological Systems. *Complexity*, **2020**, Article ID: 5476842. https://doi.org/10.1155/2020/5476842

[8] Cheneke, K.R., Rao, K.P. and Edessa, G.K. (2021) Application of a New Generalized Fractional Derivative and Rank of Control Measures on Cholera Transmission Dynamics. *International Journal of Mathematics and Mathematical Sciences*, **2021**, Article ID: 2104051. https://doi.org/10.1155/2021/2104051

[9] Zhang, L., Rahman, M.U., Ahmad, S., Riaz, M.B. and Jarad, F. (2022) Dynamics of Fractional Order Delay Model of Coronavirus Disease. *AIMS Mathematics*, **7**, 4211-4232. https://doi.org/10.3934/math.2022234

[10] Naji, F.A. and Al-Sharaa, I. (2022) Controllability of Impulsive Fractional Nonlinear Control System with Mittag-Leffler Kernel in Banach Space. *International Journal of Nonlinear Analysis and Applications*, **13**, 3257-3280.

[11] Baeumer, B., Kovács, M., Meerschaert, M.M. and Sankaranarayanan, H. (2018) Reprint of: Boundary Conditions for Fractional Diffusion. *Journal of Computational and Applied Mathematics*, **339**, 414-430. https://doi.org/10.1016/j.cam.2018.03.007

[12] Heydari, M.H., Hooshmandasl, M.R., Maalek Ghaini, F.M. and Fereidouni, F. (2013) Two-Dimensional Legendre Wavelets for Solving Fractional Poisson Equation with Dirichlet Boundary Conditions. *Engineering Analysis with Boundary Elements*, **37**, 1331-1338. https://doi.org/10.1016/j.enganabound.2013.07.002

[13] Aghdam, Y.E., Mesgrani, H., Javidi, M. and Nikan, O. (2020) A Computational Approach for the Space-Time Fractional Advection-Diffusion Equation Arising in Contaminant Transport through Porous Media. *Engineering with Computers*, **37**, 3615-3627. https://doi.org/10.1007/s00366-020-01021-y

[14] Khan, H., Khan, Q., Kumam, P., Tchier, F., Singh, G., *et al.* (2022) A Modified Approach of Adomian Decomposition Method to Solve Two-Term Diffusion Wave and Time Fractional Telegraph Equations. *IEEE Access*, **10**, 77475-77486. https://doi.org/10.1109/access.2022.3183620

[15] Al-Mazmumy, M., Alsulami, M. and Al-Yazidi, N. (2025) Algorithms and Applications of the New Modified Decomposition Method to Solve Initial-Boundary Value Problems for Fractional Partial Differential Equations. *AIMS Mathematics*, **10**, 5806-5829. https://doi.org/10.3934/math.2025267

[16] Masood, S., Khan, H., Shah, R., Mustafa, S., Khan, Q., *et al.* (2022) A New Modified Technique of Adomian Decomposition Method for Fractional Diffusion Equations with Initial-Boundary Conditions. *Journal of Function Spaces*, **2022**, Article ID: 6890517. https://doi.org/10.1155/2022/6890517

[17] El-Sayed, A.M.A., Elsaid, A., El-Kalla, I.L. and Hammad, D. (2012) A Homotopy Perturbation Technique for Solving Partial Differential Equations of Fractional Order in Finite Domains. *Applied Mathematics and Computation*, **218**, 8329-8340. https://doi.org/10.1016/j.amc.2012.01.057

[18] Daftardar-Gejji, V. and Bhalekar, S. (2010) Solving Fractional Boundary Value Problems with Dirichlet Boundary Conditions Using a New Iterative Method. *Computers & Mathematics with Applications*, **59**, 1801-1809. https://doi.org/10.1016/j.camwa.2009.08.018

[19] Bin Jebreen, H. and Cattani, C. (2022) Solving Time-Fractional Partial Differential Equation Using Chebyshev Cardinal Functions. *Axioms*, **11**, Article No. 642.

https://doi.org/10.3390/axioms11110642

[20] Aslefallah, M. and Rostamy, D. (2014) Numerical Solution for Poisson Fractional Equation via Finite Differences Theta-method. *Journal of Mathematics and Computer Science*, **12**, 132-142. https://doi.org/10.22436/jmcs.012.02.05

[21] Amin, M., Abbas, M., Iqbal, M.K. and Baleanu, D. (2020) Numerical Treatment of Time-Fractional Klein-Gordon Equation Using Redefined Extended Cubic B-Spline Functions. *Frontiers in Physics*, **8**, Article No. 288. https://doi.org/10.3389/fphy.2020.00288

[22] Adomian, G. (1994) Solving Frontier Problems of Physics: The Decomposition Method. Kluwer Academic Publishers.

[23] Wazwaz, A.M. (2009) Partial Differential Equations and Solitary Waves Theory. Springer and Higher Education Press.

[24] El-Kalla, I.L. (2010) New Results on the Analytic Summation of Adomian Series for Some Classes of Differential and Integral Equations. *Applied Mathematics and Computation*, **217**, 3756-3763. https://doi.org/10.1016/j.amc.2010.09.034

[25] Odibat, Z. and Momani, S. (2008) Numerical Methods for Nonlinear Partial Differential Equations of Fractional Order. *Applied Mathematical Modelling*, **32**, 28-39. https://doi.org/10.1016/j.apm.2006.10.025

[26] Khan, M., Hussain, M., Jafari, H., and Khan, Y. (2010) Application of Laplace de-Composition Method to Solve Nonlinear Coupled Partial Differential Equations. *World Applied Sciences Journal*, **9**, 13-19.

[27] Thabet, H. and Kendre, S. (2019) New Modification of Adomian Decomposition Method for Solving a System of Nonlinear Fractional Partial Differential Equations. *International Journal of Advances in Applied Mathematics and Mechanics*, **6**, 1-13.

[28] Cherruault, Y. and Adomian, G. (1993) Decomposition Methods: A New Proof of Convergence. *Mathematical and Computer Modelling*, **18**, 103-106. https://doi.org/10.1016/0895-7177(93)90233-o

[29] Abbaoui, K. and Cherruault, Y. (1994) Convergence of Adomian's Method Applied to Nonlinear Equations. *Mathematical and Computer Modelling*, **20**, 69-73. https://doi.org/10.1016/0895-7177(94)00163-4

[30] Li, W. and Pang, Y. (2020) Application of Adomian Decomposition Method to Nonlinear Systems. *Advances in Difference Equations*, **2020**, Article No. 67. https://doi.org/10.1186/s13662-020-2529-y

[31] El-Sayed, A.M.A. and Gaber, M. (2006) The Adomian Decomposition Method for Solving Partial Differential Equations of Fractal Order in Finite Domains. *Physics Letters A*, **359**, 175-182. https://doi.org/10.1016/j.physleta.2006.06.024

[32] Lesnic, D. (2001) A Computational Algebraic Investigation of the Decomposition Method for Time-Dependent Problems. *Applied Mathematics and Computation*, **119**, 197-206. https://doi.org/10.1016/s0096-3003(99)00257-x

[33] Ebaid, A. (2010) Modification of Lesnic's Approach and New Analytic Solutions for Some Nonlinear Second-Order Boundary Value Problems with Dirichlet Boundary Conditions. *Zeitschrift für Naturforschung A*, **65**, 692-696. https://doi.org/10.1515/zna-2010-8-910

[34] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. (2006) Theory and Applications of Fractional Differential Equations. Elsevier.

[35] Wazwaz, A.M. (1999) A Reliable Modification of Adomian Decomposition Method. *Applied Mathematics and Computation*, **102**, 77-86. https://doi.org/10.1016/s0096-3003(98)10024-3

[36] Entezari, M., Abbasbandy, S. and Babolian, E. (2019) Numerical Solution of Fractional Partial Differential Equations with Normalized Bernstein Wavelet Method. *Applications and Applied Mathematics: An International Journal*, **14**, 890-909.

[37] Xie, J., Huang, Q., Zhao, F. and Gui, H. (2017) Block Pulse Functions for Solving Fractional Poisson Type Equations with Dirichlet and Neumann Boundary Conditions. *Boundary Value Problems*, **2017**, Article No. 32. https://doi.org/10.1186/s13661-017-0766-0

[38] Shikrani, R., Hashmi, M.S., Khan, N., Ghaffar, A., Nisar, K.S., Singh, J., *et al*. (2020) An Efficient Numerical Approach for Space Fractional Partial Differential Equations. *Alexandria Engineering Journal*, **59**, 2911-2919. https://doi.org/10.1016/j.aej.2020.02.036

[39] Liu, B. and Wang, W. (2024) An Efficient Numerical Scheme in Reproducing Kernel Space for Space Fractional Partial Differential Equations. *AIMS Mathematics*, **9**, 33286-33300. https://doi.org/10.3934/math.20241588

[40] Saadatmandi, A. and Dehghan, M. (2011) A Tau Approach for Solution of the Space Fractional Diffusion Equation. *Computers & Mathematics with Applications*, **62**, 1135-1142. https://doi.org/10.1016/j.camwa.2011.04.014

[41] Azizi, H. and Loghmani, G.B. (2013) Numerical Approximation for Space Fractional Diffusion Equations via Chebyshev Finite Difference Method. *Journal of Fractional Calculus and Applications*, **4**, 303-311.

[42] Alfalqi, S., Boukhari, B., Bchatnia, A. and Beniani, A. (2024) Advanced Neural Network Approaches for Coupled Equations with Fractional Derivatives. *Boundary Value Problems*, **2024**, Article No. 96. https://doi.org/10.1186/s13661-024-01899-3