Applied Mathematics, 2026, 17(8), 477-503
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Hesitant Fuzzy Linguistic Pythagorean Fuzzy Sets and Its Application in Credit Evaluation of Enterprise

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How to cite this paper: Zhang, M. and Li, Y. (2026) Hesitant Fuzzy Linguistic Pythagorean Fuzzy Sets and Its Application in Credit Evaluation of Enterprise. *Applied Mathematics*, **17**, 477-503.
https://doi.org/10.4236/am.2026.178028

Received: July 8, 2026
Accepted: August 9, 2026
Published: August 12, 2026

Abstract

Exploring the determination method of membership degree and non-membership degree with higher richness of linguistic elicitation is an important issue in the application research of Pythagorean fuzzy sets (PFSs) in multi-attribute group decision-making (MAGDM). Firstly, the definition of hesitant fuzzy linguistic element normalized score function (HFLENSF) was proposed by using the linguistic scale function, and the mapping from hesitant fuzzy linguistic term set (HFLTS) to [0, 1] interval was realized. Based on the HFLENSF, the definition of hesitant fuzzy linguistic Pythagorean fuzzy set (HFLPFS) was then proposed. Thus, the HFLTS was reasonably introduced into the PFS. Secondly, the HFLPFS was applied to MAGDM, and a TOPSIS method for MAGDM based on HFLPFS was constructed. Finally, the proposed method was applied to the credit evaluation of the listed companies in strategic emerging industries in China, and an application example analysis was carried out. The application example analysis results show that the ranking of alternatives obtained by the proposed method is consistent with that obtained by the TOPSIS method for MAGDM based on linguistic Pythagorean fuzzy set (LPFS), but the discrimination degree of the former for alternatives is 3.8724, which is higher than 3.5188 of the latter, which proves the feasibility and effectiveness of the proposed method. This study enriches the theoretical framework of Pythagorean fuzzy sets, and expands the applicability and methodology of Pythagorean fuzzy sets in MAGDM.

Keywords

Pythagorean Fuzzy Set, Hesitant Fuzzy Linguistic Term Set, Hesitant Fuzzy Linguistic Element, Linguistic Scale Function, Multiple-Attribute Group

Decision-Making, TOPSIS Method, Enterprise Credit Evaluation

1. Introduction

Pythagorean fuzzy sets (PFSs) [1] as a new generalization of fuzzy sets (FSs) can handle uncertain information more flexibly in the process of decision making [2] and have thus gained wide application in multi-attribute group decision-making (MAGDM) [2]-[31]. However, in practical applications, Pythagorean fuzzy sets face challenges in determining membership degree and non-membership degree. Existing literatures generally employed two types of methods—quantitative and qualitative—to characterize membership degree and non-membership degree. Among them, quantitative methods mainly included: expert assignment [1] [3], hesitant fuzzy sets [2] [4] [5], interval-valued hesitant fuzzy sets [6] [7], probabilistic interval-valued hesitant fuzzy sets [8], dual hesitant fuzzy sets [9], probabilistic dual hesitant fuzzy sets [10], type-2 hesitant fuzzy sets [11], triangular fuzzy numbers [12], principal values [13], type-2 Pythagorean fuzzy sets [14], interval values [15]-[20], continuous interval values [21], rough intervals [22], etc. Qualitative methods mainly included: linguistic terms [23]-[28], two-tuple linguistic information [29] [30], linguistic hesitant fuzzy sets [31], etc. Overall, compared to quantitative methods, qualitative methods have advantages in reflecting the fuzziness and hesitation of expert decision-making. However, existing qualitative methods still fall short in terms of richness of linguistic elicitation. It is well known that the higher the richness of linguistic elicitation is, the more sufficient the opinions expressed by experts are. Therefore, it is an important topic in the application research of Pythagorean fuzzy sets in MAGDM to explore the method of determining membership degree and non-membership degree with higher richness of linguistic elicitation.

In [32], hesitation fuzzy sets [33] were extended from quantitative settings to qualitative settings, the concept of a hesitant fuzzy linguistic term set (HFLTS) was introduced to provide a linguistic and computational basis to increase the richness of linguistic elicitation based on the fuzzy linguistic approach and the use of context-free grammars by using comparative terms. The use of HFLTSs helps elicit comparative linguistic expressions (CLEs) when experts are hesitant among different linguistic terms to provide their assessments.

Based on this, some scholars attempted to introduce HFLTS into intuitionistic fuzzy set (IFS), using HFLTS to describe the membership degree and non-membership degree of an element $x \in X$ to intuitionistic fuzzy set (IFS). Beg and Rashid [34] first proposed the concept of hesitant intuitionistic fuzzy linguistic term set (HIFLTS), providing a linguistic and computational basis to manage the situations in which experts assess an alternative in possible linguistic interval and impossible linguistic interval. The follow-up studies were carried out under this theoretical framework [35]-[41]. However, due to the fact that the hesitant fuzzy linguistic element is a set composed of the elements in the linguistic term set and does not have the property of real numbers, the existing research either cannot define the ranges of membership degree and non-membership degree depicted by hesitant fuzzy linguistic elements, or the defined ranges of membership degree and non-membership degree are inconsistent with the ranges of membership degree and non-membership degree defined by orthopair fuzzy sets, which leads to bias in the defined boundary condition of the HIFLTS.

In view of this, inspired by [38], this paper utilizes the linguistic scale function, which can transform qualitative information and quantitative data, to propose the definition of the hesitant fuzzy linguistic element normalized score function (HFLENSF), realizing the mapping from hesitant fuzzy linguistic term set (HFLTS) to the [0, 1] interval; and the definition of the hesitant fuzzy linguistic Pythagorean fuzzy set (HFLPFS) is then proposed. Thus, the hesitant fuzzy linguistic term set is reasonably introduced into the Pythagorean fuzzy set. On this basis, using HFLPFS, a TOPSIS method for MAGDM is constructed and applied to the credit evaluation of the listed companies in strategic emerging industries in China.

The remaining part of this paper is structured as follows: Section 2 introduces the preparatory knowledge employed in this study; Section 3 proposes the definition of hesitant fuzzy linguistic Pythagorean fuzzy set (HFLPFS); Section 4 constructs a TOPSIS method for MAGDM based on hesitant fuzzy linguistic Pythagorean fuzzy set (HFLPFS); Section 5 describes the application example analysis results; Section 6 discusses the results obtained and Section 7 concludes this paper.

2. Preparatory Knowledge

This section briefly introduces the definition of Pythagorean Fuzzy Set, the related definitions of hesitant fuzzy linguistic term set and the definition of linguistic scale function reported in the existing studies. This is the basis of Section 3 and Section 4.

2.1. Definition of Pythagorean Fuzzy Set

Definition 1 [1]: The Pythagorean fuzzy set defined on a non-empty set X as objects having the form

$$P = \left\{ x, \mu_P(x), \nu_P(x) : x \in X \right\}$$

where the functions $\mu_P(x): X \to [0,1]$ and $\nu_P(x): X \to [0,1]$, denote the degree of membership and the degree of non-membership of each element $x \in X$ to the set P respectively, and $0 \le (\mu_P(x))^2 + (\nu_P(x))^2 \le 1$ for all $x \in X$. For any Pythagorean fuzzy set P and $x \in X$, $\pi_P(x) = \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2}$ is called the degree of indeterminacy of x to P.

2.2. Related Definitions of Hesitant Fuzzy Linguistic Term Set

Definition 2 [32] [42]: Let $X = \{x_1, x_2, \cdots, x_n\}$ be a universe of discourse, and

$S=\{s_\beta | \beta=-\tau,\cdots,-1,0,1,\cdots,\tau\}$ be a linguistic term set. A hesitant fuzzy linguistic term set (HFLTS) in X is an object having the form

$$H_S=\{\langle x_i,h_S(x_i)\rangle | x_i\in X\}$$

where $h_S(x_i)$ is the set of elements in S, called hesitant fuzzy linguistic element (HFLE), it can be expressed as $h_S(x_i)=\{s_{\varphi_l}(x_i) | s_{\varphi_l}(x_i)\in S, l=1,2,\cdots,L\}$, where $s_{\varphi_l}(x_i)$ is the φ_l-th element in $h_S(x_i)$, L denotes the number of elements in $h_S(x_i)$.

Definition 3 [42] [43]: Let G_H be a context-free grammar, and $S=\{s_\beta | \beta=-\tau,\cdots,-1,0,1,\cdots,\tau\}$ be a linguistic term set. The elements of $G_H=(V_N,V_T,I,P)$ are defined as follows:

V_N = {primary term, composite term, unary relation, binary relation, conjunction}; V_T = {"less than", "more than", "at least", "at most", "between", "and", "$s_{-\tau}$", $\cdots$, "s_{-1}", "s_0", "s_1", $\cdots$, "s_τ"}; $I\in V_N$; P = { I refers to the primary term or composite term; the primary term refers to "$s_{-\tau}$", $\cdots$, "s_{-1}", "s_0", "s_1", $\cdots$, "s_τ"; the composite term refers to unary relation + primary term, or binary relation + primary term + conjunction + primary term; the unary relation refers to "less than" or "more than" or "at least" or "at most"; the binary relation refers to "between"; the conjunction refers to "and"}.

Definition 4 [42] [43]: Let $S=\{s_\beta | \beta=-\tau,\cdots,-1,0,1,\cdots,\tau\}$ be a linguistic term set. Under the S, the linguistic expression generated by G_H is $ll\in S_{ll}$, where S_{ll} is the set of all linguistic expressions. Then the S_{ll} can be transformed into the hesitant fuzzy linguistic term set H_S by the transformation function $E_{G_H}: S_{ll}\to H_S$:

1) $E_{G_H}(s_\beta)=\{s_\beta | s_\beta\in S\}$;
2) $E_{G_H}(\text{at most } s_\alpha)=\{s_\beta | s_\beta\in S \text{ and } s_\beta\le s_\alpha\}$;
3) $E_{G_H}(\text{less than } s_\alpha)=\{s_\beta | s_\beta\in S \text{ and } s_\beta< s_\alpha\}$;
4) $E_{G_H}(\text{at least } s_\alpha)=\{s_\beta | s_\beta\in S \text{ and } s_\beta\ge s_\alpha\}$;
5) $E_{G_H}(\text{more than } s_\alpha)=\{s_\beta | s_\beta\in S \text{ and } s_\beta> s_\alpha\}$;
6) $E_{G_H}(\text{between } s_\alpha \text{ and } s_\gamma)=\{s_\beta | s_\beta\in S \text{ and } s_\alpha\le s_\beta\le s_\gamma\}$.

2.3. Definition of Linguistic Scale Function

The relationship between element s_i and its index i in the language term set $S=\{s_i | i=0,1,2,\cdots,2t\}$ is strictly monotonically increasing [44]. Based on this, [45] provides the definition of linguistic scale function:

Definition 5 [45]: If $\theta_i\in R^+\left(R^+=\{r | r\ge 0, r\in R\}\right)$ is a numerical value, the linguistic scale function f that constitutes the mapping from s_i to θ_i ($i=0,1,2,\cdots,2t$) is defined as follows:

$$f : s_i \to \theta_i \left(i = 0,1,2,\cdots,2t \right) \tag{1}$$

where $0 \le \theta_0 < \theta_1 < \cdots < \theta_{2t}$.

Obviously, function f is a strictly monotonically increasing function with respect to index i. The symbol θ_i ($i = 0,1,2,\cdots,2t$) reflects the decision maker's preference when using the linguistic term $s_i \in S$ ($i = 0,1,2,\cdots,2t$). Therefore, the function value actually represents the semantics of linguistic terms.

In [45], three possible choices for linguistic scale function are listed:

1) The definition of linguistic scale function based on index function $I\left(s_i\right) = i$ is as follows:

$$f_1\left(s_i\right) = \theta_i = i/2t \left(i = 0,1,\cdots,2t \right) \tag{2}$$

where $\theta_i \in [0,1]$.

In this case, the evaluation scale of language information is evenly divided. In addition, its form is simple and universal, but lacks a reasonable theoretical basis [46].

2) The definition of linguistic scale function based on [47] and [48] is as follows:

$$f_2\left(s_i\right) = \theta_i = \begin{cases} \dfrac{a^t - a^{t-i}}{2a^t - 2}, & i = 0,1,\cdots,t \\ \dfrac{a^t + a^{i-t} - 2}{2a^t - 2}, & i = t+1, t+2,\cdots,2t \end{cases} \tag{3}$$

In this case, as the linguistic term set extends from the middle to both ends, the absolute deviation between adjacent language subscripts increases. The value of parameter a is usually determined subjectively, with $a = \sqrt[k]{9}$ generally considered, where k represents the scale level. If $k = 7$, then $a = \sqrt[7]{9} \approx 1.37$ [47].

3) Based on [46], the improved linguistic scale function is defined as follows:

$$f_3\left(s_i\right) = \theta_i = \begin{cases} \dfrac{t^\alpha - (t-i)^\alpha}{2t^\alpha}, & i = 0,1,\cdots,t \\ \dfrac{t^\beta + (i-t)^\beta}{2t^\beta}, & i = t+1, t+2,\cdots,2t \end{cases} \tag{4}$$

where $\alpha, \beta \in [0,1]$.

In this case, as the linguistic term set extends from the middle to both ends, the absolute deviation between adjacent language subscripts will decrease. Specifically, when $2t = 6$, parameters α and β typically take values of 0.88 [38].

3. Hesitant Fuzzy Linguistic Pythagorean Fuzzy Set

This section proposes the definition of the hesitant fuzzy linguistic element normalized score function (HFLENSF), and then proposes the definition of the hesitant fuzzy linguistic Pythagorean fuzzy set (HFLPFS).

3.1. Definition of HFLENSF

Inspired by [38], according to Definitions 2 and 5, Definition 6 is proposed as follows:

Definition 6: Let $X=\{x_1,x_2,\cdots,x_n\}$ be a universe of discourse, $S=\{s_0,s_1,\cdots,s_{2t}\}$ be a linguistic term set, $H_S=\{\langle x_i,h_S(x_i)\rangle | x_i\in X, i=1,2,\cdots,n\}$ be a hesitant fuzzy linguistic term set on the domain X, where $h_S(x_i)$ is the hesitant fuzzy linguistic element, $h_S(x_i)=\{s_{\eta_l}(x_i) | s_{\eta_l}(x_i)\in S, l=1,2,\cdots,L(h_S(x_i))\}$, $s_{\eta_l}(x_i)$ is the η_l-th element in $h_S(x_i)$, and $L(h_S(x_i))$ denotes the number of elements in $h_S(x_i)$. Then the normalized score function of $h_S(x_i)$ is $\hat{s}:H_S\to[0,1]$, and its expression is

$$\hat{s}(h_S(x_i))=\sum_{s_{\eta_l}(x_i)\in h_S(x_i)}\frac{f(s_{\eta_l}(x_i))}{L(h_S(x_i))} \tag{5}$$

where $\hat{s}(h_S(x_i))$ is the normalized score of $h_S(x_i)$, $\hat{s}(h_S(x_i))\in[0,1]$; $f(\bullet)$ is the linguistic scale function [45].

According to [45], the linguistic scale function f is a strictly monotonically increasing function with respect to index i ($i=0,1,2,\cdots,2t$). Therefore, Equation (5) uses the simple average method to calculate the normalized score of $h_S(x_i)$.

Proof:

It is easy to prove that for the linguistic scale functions shown in Equation (2), Equation (3) and Equation (4), the ranges of θ_i are all $[0,1]$, therefore, we have $f(s_{\eta_l}(x_i))\in[0,1]$. When there is only one element s_0 in $h_S(x_i)$, $f(s_{\eta_l}(x_i))=0$. From Equation (5), we have $\hat{s}(h_S(x_i))=0$. When there is only one element s_{2t} in $h_S(x_i)$, $f(s_{\eta_l}(x_i))=1$. From Equation (5), we have $\hat{s}(h_S(x_i))=1$. Therefore, we have $\hat{s}(h_S(x_i))\in[0,1]$. Q.E.D.

3.2. Definition of HFLPFS

According to Definitions 1 and 6, Definition 7 is proposed as follows:

Definition 7 Let $X=\{x_1,x_2,\cdots,x_n\}$ be a universe of discourse, $S=\{s_0,s_1,\cdots,s_{2t}\}$ be a linguistic term set, and a hesitant fuzzy linguistic Pythagorean fuzzy set $P_{\hat{s}}$ in X under S is an object with the following form

$$P_{\hat{s}}=\{x_i,\hat{s}(h_S^{\mu}(x_i)),\hat{s}(h_S^{\nu}(x_i)):x_i\in X\}$$

where $h_S^{\mu}(x_i)\subset S$ is the hesitant fuzzy linguistic element characterizing the membership degree of element x_i to $P_{\hat{s}}$, it can be expressed as $h_S^{\mu}(x_i)=\{s_{\eta_l^{\mu}}(x_i) | s_{\eta_l^{\mu}}(x_i)\in S, l=1,2,\cdots,L(h_S^{\mu}(x_i))\}$, $s_{\eta_l^{\mu}}(x_i)$ is the η_l^{μ}-th element in $h_S^{\mu}(x_i)$, and $L(h_S^{\mu}(x_i))$ represents the number of elements in $h_S^{\mu}(x_i)$; $\hat{s}(h_S^{\mu}(x_i))$ is the normalized score of the hesitant fuzzy linguistic element characterizing the membership degree of element x_i to $P_{\hat{s}}$. According to Equation (5), its expression is

$$\hat{s}(h_S^{\mu}(x_i))=\sum_{s_{\eta_l^{\mu}}(x_i)\in h_S^{\mu}(x_i)}\frac{f(s_{\eta_l^{\mu}}(x_i))}{L(h_S^{\mu}(x_i))} \tag{6}$$

where $\hat{s}\left(h_S^{\mu}(x_i)\right)\in[0,1]$; $f(\bullet)$ is the linguistic scale function [45].

In addition, $h_S^{\nu}(x_i)\subset S$ is the hesitant fuzzy linguistic element characterizing the non-membership degree of element x_i to $P_{\hat{s}}$, it can be expressed as $h_S^{\nu}(x_i)=\left\{s_{\eta_l^{\nu}}(x_i)\middle|s_{\eta_l^{\nu}}(x_i)\in S, l=1,2,\cdots,L\left(h_S^{\nu}(x_i)\right)\right\}$, $s_{\eta_l^{\nu}}(x_i)$ is the η_l^{ν}-th element in $h_S^{\nu}(x_i)$, and $L\left(h_S^{\nu}(x_i)\right)$ represents the number of elements in $h_S^{\nu}(x_i)$; $\hat{s}\left(h_S^{\nu}(x_i)\right)$ is the normalized score of the hesitant fuzzy linguistic element characterizing the non-membership degree of element x_i to $P_{\hat{s}}$. According to Equation (5), its expression is

$$\hat{s}\left(h_S^{\nu}(x_i)\right)=\sum_{s_{\eta_l^{\nu}}(x_i)\in h_S^{\nu}(x_i)}\frac{f\left(s_{\eta_l^{\nu}}(x_i)\right)}{L\left(h_S^{\nu}(x_i)\right)} \tag{7}$$

where $\hat{s}\left(h_S^{\nu}(x_i)\right)\in[0,1]$; $f(\bullet)$ is the linguistic scale function [45].

For all $x_i\in X$, $\hat{s}\left(h_S^{\mu}(x_i)\right)$ and $\hat{s}\left(h_S^{\nu}(x_i)\right)$ satisfy the boundary condition: $0\le\left(\hat{s}\left(h_S^{\mu}(x_i)\right)\right)^2+\left(\hat{s}\left(h_S^{\nu}(x_i)\right)\right)^2\le 1$.

For any $P_{\hat{s}}$ and $x_i\in X$, $\hat{s}\left(h_S^{\pi}(x_i)\right)=\sqrt{1-\left(\hat{s}\left(h_S^{\mu}(x_i)\right)\right)^2-\left(\hat{s}\left(h_S^{\nu}(x_i)\right)\right)^2}$ is identified as the normalized score of the hesitant fuzzy linguistic element characterizing the indeterminacy degree of x_i to $P_{\hat{s}}$.

For simplicity, when there is only one element in X, the shorthand $P_{\hat{s}}=\left\{\hat{s}\left(h_S^{\mu}\right),\hat{s}\left(h_S^{\nu}\right)\right\}$ is called the hesitant fuzzy linguistic Pythagorean fuzzy number.

4. A TOPSIS Method for MAGDM Based on HFLPFS

This section utilizes the hesitant fuzzy linguistic Pythagorean fuzzy set (HFLPFS) proposed in Section 3 to construct a TOPSIS method for MAGDM.

4.1. Problem Description

Let $A=(A_1,A_2,\cdots,A_m)$ be a finite set of alternatives A_i ($i=1,2,\cdots,m$), and $C=(C_1,C_2,\cdots,C_n)$ be a set of attributes C_j ($j=1,2,\cdots,n$) to compare the alternatives; Let $A^*=\{1,2,\cdots,m\}$ be a set of subscripts of alternatives, and $C^*=\{1,2,\cdots,n\}$ be a set of subscripts of attributes. The weight vector of attributes is $W_C=(w_1,w_2,\cdots,w_n)$, where $w_i\in[0,1]$, and $\sum_{i=1}^{n}w_i=1$. Let $D=(D_1,D_2,\cdots,D_k)$ be a set of experts D_q ($q=1,2,\cdots,k$), and $D^*=\{1,2,\cdots,k\}$ be a set of subscripts of experts. The weight vector of experts is $W_D=(w_1,w_2,\cdots,w_k)$, where $w_q\in[0,1]$, and $\sum_{q=1}^{k}w_q=1$.

4.2. Decision-Making Steps

The specific steps of the TOPSIS method for MAGDM based on HFLPFS are as follows:

Step 1: Establish a linguistic term set S for characterizing the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j.

Let $S=\{s_\beta \mid \beta=0,1,\cdots,\tau\}$ be the linguistic term set, where $s_0,s_1,\cdots,s_\tau$ is the linguistic term for characterizing the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j; the larger β is, the higher the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j is; the smaller β is, the lower the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j is. For example, when the linguistic term set S = { s_0 = extremely low, s_1 = very low, s_2 = low, s_3 = medium, s_4 = high, s_5 = very high, s_6 = extremely high}, where the linguistic term s_0 means that the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j is extremely low, et cetera.

Step 2: Establish the linguistic expression initial decision matrix of the expert D_q.

According to Definitions 3 and 4, under the linguistic term set S, the expert D_q employs the linguistic expressions $ll \in S_{ll}$ generated by the context-free grammar G_H (see Definition 3) to evaluate the membership degree μ_{ij} and non-membership degree ν_{ij} of alternative A_i with respect to attribute C_j. Thus, the linguistic expression initial decision matrix of the expert D_q ($q=1,2,\cdots,k$) can be obtained as follows:

$$\begin{bmatrix} \left(ll^{\mu_{11}(q)}, ll^{\nu_{11}(q)}\right) & \left(ll^{\mu_{12}(q)}, ll^{\nu_{12}(q)}\right) & \cdots & \left(ll^{\mu_{1n}(q)}, ll^{\nu_{1n}(q)}\right) \\ \left(ll^{\mu_{21}(q)}, ll^{\nu_{21}(q)}\right) & \left(ll^{\mu_{22}(q)}, ll^{\nu_{22}(q)}\right) & \cdots & \left(ll^{\mu_{2n}(q)}, ll^{\nu_{2n}(q)}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(ll^{\mu_{m1}(q)}, ll^{\nu_{m1}(q)}\right) & \left(ll^{\mu_{m2}(q)}, ll^{\nu_{m2}(q)}\right) & \cdots & \left(ll^{\mu_{mn}(q)}, ll^{\nu_{mn}(q)}\right) \end{bmatrix}$$

where $ll^{\mu_{ij}(q)}$ is the linguistic expression used by expert D_q to evaluate the membership degree μ_{ij} of alternative A_i with respect to attribute C_j; $ll^{\nu_{ij}(q)}$ is the linguistic expression used by expert D_q to evaluate the non-membership degree ν_{ij} of alternative A_i with respect to attribute C_j.

Step 3: Establish the hesitant fuzzy linguistic initial decision matrix of the expert D_q.

According to Definitions 4, through the transformation function $E_{G_H}: S_{ll} \to H_S^{ij}$, the S_{ll} is further transformed into the hesitant fuzzy linguistic term set H_S^{ij}. Thus, the hesitant fuzzy linguistic initial decision matrix of the expert D_q ($q=1,2,\cdots,k$) can be obtained as follows:

$$\begin{bmatrix} \left(h_S^{\mu_{11}(q)}, h_S^{\nu_{11}(q)}\right) & \left(h_S^{\mu_{12}(q)}, h_S^{\nu_{12}(q)}\right) & \cdots & \left(h_S^{\mu_{1n}(q)}, h_S^{\nu_{1n}(q)}\right) \\ \left(h_S^{\mu_{21}(q)}, h_S^{\nu_{21}(q)}\right) & \left(h_S^{\mu_{22}(q)}, h_S^{\nu_{22}(q)}\right) & \cdots & \left(h_S^{\mu_{2n}(q)}, h_S^{\nu_{2n}(q)}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(h_S^{\mu_{m1}(q)}, h_S^{\nu_{m1}(q)}\right) & \left(h_S^{\mu_{m2}(q)}, h_S^{\nu_{m2}(q)}\right) & \cdots & \left(h_S^{\mu_{mn}(q)}, h_S^{\nu_{mn}(q)}\right) \end{bmatrix}$$

where $h_S^{\mu_{ij}(q)} = \left\{ s_{\eta_l^{\mu_{ij}(q)}} \middle| s_{\eta_l^{\mu_{ij}(q)}} \in S, l = 1, 2, \cdots, L\left(h_S^{\mu_{ij}(q)}\right) \right\}$ is the hesitant fuzzy linguistic element used by expert D_q to evaluate the membership degree μ_{ij} of alternative A_i with respect to attribute C_j, $s_{\eta_l^{\mu_{ij}(q)}}$ is the $\eta_l^{\mu_{ij}(q)}$-th element in $h_S^{\mu_{ij}(q)}$, and $L\left(h_S^{\mu_{ij}(q)}\right)$ represents the number of elements in $h_S^{\mu_{ij}(q)}$; $h_S^{\nu_{ij}(q)} = \left\{ s_{\eta_l^{\nu_{ij}(q)}} \middle| s_{\eta_l^{\nu_{ij}(q)}} \in S, l = 1, 2, \cdots, L\left(h_S^{\nu_{ij}(q)}\right) \right\}$ is the hesitant fuzzy linguistic element used by expert D_q to evaluate the non-membership degree ν_{ij} of alternative A_i with respect to attribute C_j, $s_{\eta_l^{\nu_{ij}(q)}}$ is the $\eta_l^{\nu_{ij}(q)}$-th element in $h_S^{\nu_{ij}(q)}$, and $L\left(h_S^{\nu_{ij}(q)}\right)$ represents the number of elements in $h_S^{\nu_{ij}(q)}$.

Step 4: Calculate the hesitant fuzzy linguistic orthopair fuzzy decision matrix of the expert D_q.

According to Definition 5, Equation (2) lacks a reasonable theoretical basis, while in Equation (3), as the linguistic term set extends from the middle to both ends, the absolute deviation between adjacent language subscripts increases. Therefore, we select Equation (4) as the specific linguistic scale function. In addition, based on the experience provided by [38], we set the values of parameters α and β to 0.88.

According to Definition 6, from Equation (5), the normalized score of the hesitant fuzzy linguistic element $h_S^{\mu_{ij}(q)}$ can be calculated as

$$\hat{s}\left(h_S^{\mu_{ij}(q)}\right) = \sum_{s_{\eta_l^{\mu_{ij}(q)}} \in h_S^{\mu_{ij}(q)}} \frac{f_3\left(s_{\eta_l^{\mu_{ij}(q)}}\right)}{L\left(h_S^{\mu_{ij}(q)}\right)} \tag{8}$$

where $\hat{s}\left(h_S^{\mu_{ij}(q)}\right) \in [0,1]$.

From Equation (5), the normalized score of the hesitant fuzzy linguistic element $h_S^{\nu_{ij}(q)}$ can be calculated as

$$\hat{s}\left(h_S^{\nu_{ij}(q)}\right) = \sum_{s_{\eta_l^{\nu_{ij}(q)}} \in h_S^{\nu_{ij}(q)}} \frac{f_3\left(s_{\eta_l^{\nu_{ij}(q)}}\right)}{L\left(h_S^{\nu_{ij}(q)}\right)} \tag{9}$$

where $\hat{s}\left(h_S^{\nu_{ij}(q)}\right) \in [0,1]$.

Thus, the hesitant fuzzy linguistic orthopair fuzzy decision matrix of the expert D_q ($q = 1, 2, \cdots, k$) can be obtained as follows:

$$\begin{bmatrix} \left(\hat{s}\left(h_S^{\mu_{11}(q)}\right), \hat{s}\left(h_S^{\nu_{11}(q)}\right)\right) & \left(\hat{s}\left(h_S^{\mu_{12}(q)}\right), \hat{s}\left(h_S^{\nu_{12}(q)}\right)\right) & \cdots & \left(\hat{s}\left(h_S^{\mu_{1n}(q)}\right), \hat{s}\left(h_S^{\nu_{1n}(q)}\right)\right) \\ \left(\hat{s}\left(h_S^{\mu_{21}(q)}\right), \hat{s}\left(h_S^{\nu_{21}(q)}\right)\right) & \left(\hat{s}\left(h_S^{\mu_{22}(q)}\right), \hat{s}\left(h_S^{\nu_{22}(q)}\right)\right) & \cdots & \left(\hat{s}\left(h_S^{\mu_{2n}(q)}\right), \hat{s}\left(h_S^{\nu_{2n}(q)}\right)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\hat{s}\left(h_S^{\mu_{m1}(q)}\right), \hat{s}\left(h_S^{\nu_{m1}(q)}\right)\right) & \left(\hat{s}\left(h_S^{\mu_{m2}(q)}\right), \hat{s}\left(h_S^{\nu_{m2}(q)}\right)\right) & \cdots & \left(\hat{s}\left(h_S^{\mu_{mn}(q)}\right), \hat{s}\left(h_S^{\nu_{mn}(q)}\right)\right) \end{bmatrix}$$

where $\left(\hat{s}\left(h_S^{\mu_{ij}(q)}\right),\hat{s}\left(h_S^{\nu_{ij}(q)}\right)\right)$ is the hesitant fuzzy linguistic orthopair fuzzy number of the expert D_q.

Step 5: Establish the hesitant fuzzy linguistic Pythagorean fuzzy decision matrix of the expert D_q.

According to Definition 7, from the hesitant fuzzy linguistic orthopair fuzzy decision matrix of the expert D_q ($q=1,2,\cdots,k$), we calculate the $\left(\hat{s}\left(h_S^{\mu_{ij}(q)}\right)\right)^2+\left(\hat{s}\left(h_S^{\nu_{ij}(q)}\right)\right)^2$, $i=1,2,\cdots,m$, $j=1,2,\cdots,n$.

If $0\le\left(\hat{s}\left(h_S^{\mu_{ij}(q)}\right)\right)^2+\left(\hat{s}\left(h_S^{\nu_{ij}(q)}\right)\right)^2\le 1$, then $\hat{s}\left(h_S^{\mu_{ij}(q)}\right)$ and $\hat{s}\left(h_S^{\nu_{ij}(q)}\right)$ satisfy the boundary condition of Pythagorean fuzzy set, and the hesitant fuzzy linguistic orthopair fuzzy decision matrix of the expert D_q is also the hesitant fuzzy linguistic Pythagorean fuzzy decision matrix of the expert D_q.

If $\left(\hat{s}\left(h_S^{\mu_{ij}(q)}\right)\right)^2+\left(\hat{s}\left(h_S^{\nu_{ij}(q)}\right)\right)^2>1$, then we repeat Steps 2-4 until $\hat{s}\left(h_S^{\mu_{ij}(q)}\right)$ and $\hat{s}\left(h_S^{\nu_{ij}(q)}\right)$ satisfy the boundary condition of Pythagorean fuzzy set.

Thus, the hesitant fuzzy linguistic Pythagorean fuzzy decision matrix of the expert D_q ($q=1,2,\cdots,k$) can be obtained as follows:

$$\begin{bmatrix}
\left(\hat{s}\left(h_S^{\mu_{P,11}(q)}\right),\hat{s}\left(h_S^{\nu_{P,11}(q)}\right)\right) & \left(\hat{s}\left(h_S^{\mu_{P,12}(q)}\right),\hat{s}\left(h_S^{\nu_{P,12}(q)}\right)\right) & \cdots & \left(\hat{s}\left(h_S^{\mu_{P,1n}(q)}\right),\hat{s}\left(h_S^{\nu_{P,1n}(q)}\right)\right)\\
\left(\hat{s}\left(h_S^{\mu_{P,21}(q)}\right),\hat{s}\left(h_S^{\nu_{P,21}(q)}\right)\right) & \left(\hat{s}\left(h_S^{\mu_{P,22}(q)}\right),\hat{s}\left(h_S^{\nu_{P,22}(q)}\right)\right) & \cdots & \left(\hat{s}\left(h_S^{\mu_{P,2n}(q)}\right),\hat{s}\left(h_S^{\nu_{P,2n}(q)}\right)\right)\\
\vdots & \vdots & \ddots & \vdots\\
\left(\hat{s}\left(h_S^{\mu_{P,m1}(q)}\right),\hat{s}\left(h_S^{\nu_{P,m1}(q)}\right)\right) & \left(\hat{s}\left(h_S^{\mu_{P,m2}(q)}\right),\hat{s}\left(h_S^{\nu_{P,m2}(q)}\right)\right) & \cdots & \left(\hat{s}\left(h_S^{\mu_{P,mn}(q)}\right),\hat{s}\left(h_S^{\nu_{P,mn}(q)}\right)\right)
\end{bmatrix}$$

where $\left(\hat{s}\left(h_S^{\mu_{P,ij}(q)}\right),\hat{s}\left(h_S^{\nu_{P,ij}(q)}\right)\right)$ is the hesitant fuzzy linguistic Pythagorean fuzzy number of the expert D_q.

Step 6: Calculate the hesitant fuzzy linguistic Pythagorean fuzzy decision matrix aggregating k experts' opinions.

Drawing on [49], we give the definition of the hesitant fuzzy linguistic Pythagorean fuzzy numbers weighted geometric aggregation operator (HFLPFNWG).

Definition 8: Let $P_{\hat{s}}^{(q)}=\left\{\hat{s}\left(h_S^{\mu_P(q)}\right),\hat{s}\left(h_S^{\nu_P(q)}\right)\right\}$ ($q=1,2,\cdots,k$) be a group of hesitant fuzzy linguistic Pythagorean fuzzy numbers with weight vector $\left(w_1,w_2,\cdots,w_k\right)$, where $w_q\in[0,1]$ and $\sum_{q=1}^{k}w_q=1$. Let $\text{HFLPFNWG}:\Theta^k\to\Theta$, if

$$\text{HFLPFNWG}\left(P_{\hat{s}}^{(1)},P_{\hat{s}}^{(2)},\cdots,P_{\hat{s}}^{(k)}\right)=\left(\prod_{q=1}^{k}\left(\hat{s}\left(h_S^{\mu_P(q)}\right)\right)^{w_q},\sqrt{1-\prod_{q=1}^{k}\left(1-\left(\hat{s}\left(h_S^{\nu_P(q)}\right)\right)^2\right)^{w_q}}\right) \quad (10)$$

then HFLPFNWG is called the hesitant fuzzy linguistic Pythagorean fuzzy numbers weighted geometric aggregation operator.

Using the HFLPFNWG to aggregate k hesitant fuzzy linguistic Pythagorean fuzzy numbers $\left(\hat{s}\left(h_S^{\mu_{P,ij}(q)}\right), \hat{s}\left(h_S^{\nu_{P,ij}(q)}\right)\right)$ ($q = 1, 2, \cdots, k$), the hesitant fuzzy linguistic Pythagorean fuzzy decision matrix aggregating k experts' opinions can be obtained as follows:

$$\begin{bmatrix} \left(\hat{s}\left(h_S^{\mu_{P,11}}\right), \hat{s}\left(h_S^{\nu_{P,11}}\right)\right) & \left(\hat{s}\left(h_S^{\mu_{P,12}}\right), \hat{s}\left(h_S^{\nu_{P,12}}\right)\right) & \cdots & \left(\hat{s}\left(h_S^{\mu_{P,1n}}\right), \hat{s}\left(h_S^{\nu_{P,1n}}\right)\right) \\ \left(\hat{s}\left(h_S^{\mu_{P,21}}\right), \hat{s}\left(h_S^{\nu_{P,21}}\right)\right) & \left(\hat{s}\left(h_S^{\mu_{P,22}}\right), \hat{s}\left(h_S^{\nu_{P,22}}\right)\right) & \cdots & \left(\hat{s}\left(h_S^{\mu_{P,2n}}\right), \hat{s}\left(h_S^{\nu_{P,2n}}\right)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\hat{s}\left(h_S^{\mu_{P,m1}}\right), \hat{s}\left(h_S^{\nu_{P,m1}}\right)\right) & \left(\hat{s}\left(h_S^{\mu_{P,m2}}\right), \hat{s}\left(h_S^{\nu_{P,m2}}\right)\right) & \cdots & \left(\hat{s}\left(h_S^{\mu_{P,mn}}\right), \hat{s}\left(h_S^{\nu_{P,mn}}\right)\right) \end{bmatrix}$$

where $\left(\hat{s}\left(h_S^{\mu_{P,ij}}\right), \hat{s}\left(h_S^{\nu_{P,ij}}\right)\right)$ is the hesitant fuzzy linguistic Pythagorean fuzzy number aggregating k experts' opinions.

Step 7: Determine the hesitant fuzzy linguistic Pythagorean fuzzy positive ideal solution S^+ and the hesitant fuzzy linguistic Pythagorean fuzzy negative ideal solution S^-.

According to the hesitant fuzzy linguistic Pythagorean fuzzy decision matrix aggregating k experts' opinions, we calculate the score of $\left(\hat{s}\left(h_S^{\mu_{P,ij}}\right), \hat{s}\left(h_S^{\nu_{P,ij}}\right)\right)$ [50]: $\hat{S}\left(P_{\hat{s}}^{ij}\right) = \left(\hat{s}\left(h_S^{\mu_{P,ij}}\right)\right)^2 - \left(\hat{s}\left(h_S^{\nu_{P,ij}}\right)\right)^2$. Then the hesitant fuzzy linguistic Pythagorean fuzzy positive ideal solution S^+ is

$$\begin{aligned} S^+ &= \begin{cases} \max\limits_i \hat{S}\left(P_{\hat{s}}^{ij}\right) \mid j = 1, 2, \cdots, n, \text{ if } C_j \text{ is a benefit attribute} \\ \min\limits_i \hat{S}\left(P_{\hat{s}}^{ij}\right) \mid j = 1, 2, \cdots, n, \text{ if } C_j \text{ is a cost attribute} \end{cases} \\ &= \left\{\left(\hat{s}\left(h_S^{\mu_P}\right)_1^+, \hat{s}\left(h_S^{\nu_P}\right)_1^+\right), \left(\hat{s}\left(h_S^{\mu_P}\right)_2^+, \hat{s}\left(h_S^{\nu_P}\right)_2^+\right), \cdots, \left(\hat{s}\left(h_S^{\mu_P}\right)_n^+, \hat{s}\left(h_S^{\nu_P}\right)_n^+\right)\right\} \end{aligned} \tag{11}$$

The hesitant fuzzy linguistic Pythagorean fuzzy negative ideal solution S^- is

$$\begin{aligned} S^- &= \begin{cases} \min\limits_i \left\langle \hat{S}\left(P_{\hat{s}}^{ij}\right)\right\rangle \mid j = 1, 2, \cdots, n, \text{ if } C_j \text{ is a benefit attribute} \\ \max\limits_i \left\langle \hat{S}\left(P_{\hat{s}}^{ij}\right)\right\rangle \mid j = 1, 2, \cdots, n, \text{ if } C_j \text{ is a cost attribute} \end{cases} \\ &= \left\{\left(\hat{s}\left(h_S^{\mu_P}\right)_1^-, \hat{s}\left(h_S^{\nu_P}\right)_1^-\right), \left(\hat{s}\left(h_S^{\mu_P}\right)_2^-, \hat{s}\left(h_S^{\nu_P}\right)_2^-\right), \cdots, \left(\hat{s}\left(h_S^{\mu_P}\right)_n^-, \hat{s}\left(h_S^{\nu_P}\right)_n^-\right)\right\} \end{aligned} \tag{12}$$

Step 8: Calculate the distance $D\left(A_i, S^+\right)$ between the alternative A_i and the hesitant fuzzy linguistic Pythagorean fuzzy positive ideal solution S^+, and the distance $D\left(A_i, S^-\right)$ between the alternative A_i and the hesitant fuzzy linguistic Pythagorean fuzzy negative ideal solution S^-.

According to Definition 1 and [50], the distance between the alternative A_i and the hesitant fuzzy linguistic Pythagorean fuzzy positive ideal solution S^+ is

$$\begin{aligned} &D\left(A_i, S^+\right) \\ &= \sum_{j=1}^n w_j d\left(C_j\left(A_i\right), C_j\left(S^+\right)\right) \\ &= \sum_{j=1}^n w_j \sqrt{\frac{1}{2}\left[\left(\left(\hat{s}\left(h_S^{\mu_{P,ij}}\right)\right)^2 - \left(\hat{s}\left(h_S^{\mu_P}\right)_j^+\right)^2\right)^2 + \left(\left(\hat{s}\left(h_S^{\nu_{P,ij}}\right)\right)^2 - \left(\hat{s}\left(h_S^{\nu_P}\right)_j^+\right)^2\right)^2 + \left(\left(\hat{s}\left(h_S^{\pi_{P,ij}}\right)\right)^2 - \left(\hat{s}\left(h_S^{\pi_P}\right)_j^+\right)^2\right)^2\right]} \\ &i = 1, 2, \cdots, m \end{aligned} \tag{13}$$

The distance between the alternative A_i and the hesitant fuzzy linguistic Pythagorean fuzzy negative ideal solution S^- is

$$D\left(A_i,S^-\right)$$
$$=\sum_{j=1}^{n} w_j d\left(C_j\left(A_i\right),C_j\left(S^-\right)\right)$$
$$=\sum_{j=1}^{n} w_j \sqrt{\frac{1}{2}\left[\left(\left(\hat{s}\left(h_S^{\mu_P,ij}\right)\right)^2-\left(\hat{s}\left(h_S^{\mu_P}\right)_j^-\right)^2\right)^2+\left(\left(\hat{s}\left(h_S^{\nu_P,ij}\right)\right)^2-\left(\hat{s}\left(h_S^{\nu_P}\right)_j^-\right)^2\right)^2+\left(\left(\hat{s}\left(h_S^{\pi_P,ij}\right)\right)^2-\left(\hat{s}\left(h_S^{\pi_P}\right)_j^-\right)^2\right)^2\right]} \quad (14)$$
$$i=1,2,\cdots,m$$

Step 9: Calculate the relative closeness $RC\left(A_i\right)$ of the alternative A_i to the hesitant fuzzy linguistic Pythagorean fuzzy positive ideal solution S^+ and the ranking of alternatives.

According to [50], the relative closeness of the alternative A_i to the hesitant fuzzy linguistic Pythagorean fuzzy positive ideal solution S^+ is

$$RC\left(A_i\right)=\frac{D\left(A_i,S^-\right)}{D\left(A_i,S^-\right)+D\left(A_i,S^+\right)} \quad (15)$$

By ranking the relative closeness $RC\left(A_i\right)$ of the alternative A_i ($i=1,2,\cdots,m$) from large to small, the ranking of m alternatives A_i ($i=1,2,\cdots,m$) can be obtained. The larger the relative closeness $RC\left(A_i\right)$ is, the better the alternative A_i is. The smaller the relative closeness $RC\left(A_i\right)$ is, the worse the alternative A_i is.

5. Application Example

This section takes the credit evaluation of the listed companies in strategic emerging industries in China as an application example to illustrate the feasibility and effectiveness of the proposed method (see Section 4), and the applicability of the HFLPFSs in MAGDM is then proved.

5.1. Sample and Indicator Data

Because the purpose of the case study is to illustrate the methodology, we selected four representative manufacturing listed companies from the sample companies of the China's Strategic Emerging Industries Comprehensive Index (000891) to form an alternative set $A=\left\{A_1,A_2,A_3,A_4\right\}$, where A_1 was Zhejiang Dahua Technology Co., Ltd. (002236, Computer, communication, and other electronic equipment manufacturing industry); A_2 was Contemporary Amperex Technology Co., Ltd. (300750, Electrical machinery and equipment manufacturing industry); A_3 was Avic Xi'an Aircraft Industry Group Company Ltd. (000768, Railway, ship, aerospace and other transportation equipment manufacturing industry); A_4 was Shenzhen Green Eco-Manufacture Hi-Tech Co., Ltd. (002340, Comprehensive utilization of abandoned resources industry). For simplicity, based on [51], five indicators (attributes) were selected to evaluate the credit status of sample enterprises: Current Ratio C_1 (reflecting short-term solvency); Asset-Liability Ratio C_2

(reflecting long-term solvency); Inventory Turnover Ratio C_3 (reflecting operational capability); Return on Equity (ROE) C_4 (reflecting profitability); Net Profit Growth Rate C_5 (reflecting growth capability). Among them, C_2 is a cost-type indicator, and the others are benefit-type indicators. The sample period was set as 2023. The data for the five indicators were calculated based on the annual report data of listed companies, and the calculation results were shown in Table 1.

Table 1. Indicator data.

Indicators (Unit)	A_1	A_2	A_3	A_4
C_1 (−)	2.5176	1.5672	1.0443	1.1068
C_2 (%)	32.14	69.34	75.85	58.76
C_3 (time)	4.5950	5.7429	1.5785	3.7113
C_4 (%)	22.43	24.04	5.22	4.93
C_5 (%)	230.49	39.76	64.41	−12.79

Note: "−" indicates the indicator is unitless.

Based on Table 1, the entropy weight method [52] was used to determine the indicator weights, and the weight vector of the indicators was obtained as $W_C = (0.2439, 0.1947, 0.1263, 0.2503, 0.1848)$. In addition, there were three experts: D_1, D_2 and D_3. The cyclic mutual evaluation method [53] was used to determine the weights of the experts, and the weight vector of the experts was obtained as $W_D = (0.3976, 0.3012, 0.3012)$ (Note: The detailed calculation process see Appendix A in [54]).

5.2. Enterprise Credit Evaluation Process and Results

Step 1: Established a linguistic term set S for characterizing the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j.

According to step 1 in Section 4.2, a linguistic term set S was established to characterize the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j, $S = \{ s_0$ = extremely low, s_1 = very low, s_2 = low, s_3 = medium, s_4 = high, s_5 = very high, s_6 = extremely high}, where the linguistic term s_0 meant that the membership degree μ_{ij} (or non-membership degree ν_{ij}) of alternative A_i with respect to attribute C_j was extremely low, et cetera.

Step 2: Established the linguistic expression initial decision matrix of the expert D_q.

According to step 2 in Section 4.2, under the linguistic term set S, the expert D_q employed the linguistic expressions $ll \in S_{ll}$ generated by the context-free grammar G_H (see Definition 3) to evaluate the membership degree μ_{ij} and non-membership degree ν_{ij} of alternative A_i with respect to attribute C_j. Thus, the linguistic expression initial decision matrix of the expert D_q ($q = 1, 2, 3$) was obtained (see **Appendix A**).

Step 3: Established the hesitant fuzzy linguistic initial decision matrix of the expert D_q.

According to step 3 in Section 4.2, through the transformation function $E_{G_H}: S_{ll} \to H_S^{ij}$ (see Definition 4), the S_{ll} was further transformed into the hesitant fuzzy linguistic term set H_S^{ij}. Thus, the hesitant fuzzy linguistic initial decision matrix of the expert D_q ($q = 1,2,3$) was obtained, as shown in Table 2, Table 3 and Table 4 respectively.

Table 2. Hesitant fuzzy linguistic initial decision matrix of the expert D_1.

C_j	A_1	A_2	A_3	A_4
C_1	$(\{s_4, s_5\}, \{s_1, s_2\})$	$(\{s_1, s_2, s_3\}, \{s_4, s_5\})$	$(\{s_0, s_1\}, \{s_4, s_5, s_6\})$	$(\{s_1, s_2\}, \{s_3, s_4, s_5\})$
C_2	$(\{s_2, s_3\}, \{s_4, s_5, s_6\})$	$(\{s_2, s_3\}, \{s_4, s_5\})$	$(\{s_4, s_5\}, \{s_0, s_1, s_2\})$	$(\{s_4, s_5\}, \{s_1, s_2\})$
C_3	$(\{s_3, s_4\}, \{s_1, s_2\})$	$(\{s_0, s_1, s_2, s_3, s_4, s_5\}, \{s_1, s_2, s_3\})$	$(\{s_1, s_2\}, \{s_4, s_5, s_6\})$	$(\{s_3\}, \{s_2, s_3, s_4\})$
C_4	$(\{s_4, s_5, s_6\}, \{s_1, s_2\})$	$(\{s_5, s_6\}, \{s_0, s_1, s_2\})$	$(\{s_0, s_1, s_2\}, \{s_3, s_4, s_5, s_6\})$	$(\{s_0, s_1\}, \{s_5, s_6\})$
C_5	$(\{s_5, s_6\}, \{s_0, s_1\})$	$(\{s_3\}, \{s_2, s_3, s_4\})$	$(\{s_5, s_6\}, \{s_0, s_1, s_2\})$	$(\{s_0\}, \{s_5, s_6\})$

Table 3. Hesitant fuzzy linguistic initial decision matrix of the expert D_2.

C_j	A_1	A_2	A_3	A_4
C_1	$(\{s_4, s_5, s_6\}, \{s_0, s_1, s_2\})$	$(\{s_2, s_3\}, \{s_4, s_5\})$	$(\{s_1, s_2\}, \{s_4, s_5\})$	$(\{s_2\}, \{s_4, s_5\})$
C_2	$(\{s_2, s_3\}, \{s_4, s_5, s_6\})$	$(\{s_1, s_2, s_3\}, \{s_4, s_5\})$	$(\{s_4, s_5\}, \{s_1, s_2\})$	$(\{s_4, s_5, s_6\}, \{s_0, s_1\})$
C_3	$(\{s_2, s_3, s_4\}, \{s_3, s_4, s_5\})$	$(\{s_3, s_4, s_5\}, \{s_1, s_2, s_3\})$	$(\{s_0, s_1, s_2, s_3\}, \{s_4\})$	$(\{s_2, s_3\}, \{s_3, s_4\})$
C_4	$(\{s_4, s_5, s_6\}, \{s_0, s_1\})$	$(\{s_4, s_5\}, \{s_0, s_1, s_2\})$	$(\{s_0, s_1, s_2\}, \{s_5, s_6\})$	$(\{s_0, s_1\}, \{s_5\})$
C_5	$(\{s_5, s_6\}, \{s_0, s_1\})$	$(\{s_3, s_4\}, \{s_2\})$	$(\{s_5, s_6\}, \{s_0, s_1, s_2\})$	$(\{s_0\}, \{s_5, s_6\})$

Table 4. Hesitant fuzzy linguistic initial decision matrix of the expert D_3.

C_j	A_1	A_2	A_3	A_4
C_1	$(\{s_4, s_5, s_6\}, \{s_0, s_1, s_2\})$	$(\{s_2, s_3\}, \{s_4, s_5\})$	$(\{s_0, s_1, s_2\}, \{s_5, s_6\})$	$(\{s_1, s_2\}, \{s_4, s_5\})$
C_2	$(\{s_2, s_3\}, \{s_4, s_5\})$	$(\{s_0, s_1, s_2, s_3\}, \{s_4, s_5\})$	$(\{s_4, s_5\}, \{s_1, s_2\})$	$(\{s_5\}, \{s_0, s_1\})$
C_3	$(\{s_3, s_4\}, \{s_4, s_5\})$	$(\{s_0, s_1, s_2, s_3, s_4\}, \{s_2, s_3\})$	$(\{s_0, s_1\}, \{s_5, s_6\})$	$(\{s_2, s_3\}, \{s_4, s_5\})$
C_4	$(\{s_3, s_4\}, \{s_2, s_3\})$	$(\{s_3, s_4, s_5\}, \{s_1, s_2\})$	$(\{s_1, s_2\}, \{s_4, s_5\})$	$(\{s_0, s_1, s_2\}, \{s_5, s_6\})$
C_5	$(\{s_5\}, \{s_0, s_1\})$	$(\{s_3, s_4\}, \{s_2\})$	$(\{s_4, s_5, s_6\}, \{s_0, s_1, s_2\})$	$(\{s_0, s_1\}, \{s_5, s_6\})$

Step 4: Calculated the hesitant fuzzy linguistic orthopair fuzzy decision matrix of the expert D_q.

According to step 4 in Section 4.2, we selected Eq. (4) as the specific linguistic scale function. For the linguistic term set S, there was $2t = 6$, then the values of α and β in Equation (4) were both 0.88. Thus, the linguistic scale values θ_i ($i = 0,1,\cdots,6$) of the seven linguistic terms in S were calculated, as shown in Table 5. Taking $i = 0$ as an example, the linguistic scale value θ_0 was calculated as follows:

$$\theta_0 = \frac{3^{0.88} - (3-0)^{0.88}}{2 \times 3^{0.88}} = 0.$$

Table 5. Linguistic scale values.

i	0	1	2	3	4	5	6
θ_i	0	0.15	0.3097	0.5	0.6901	0.8498	1

Based on Table 2, Table 3, Table 4 and Table 5, the normalized score $\hat{s}\left(h_S^{\mu_{ij}(q)}\right)$ of hesitant fuzzy linguistic element $h_S^{\mu_{ij}(q)}$ was calculated by using Equation (8), and the normalized score $\hat{s}\left(h_S^{\nu_{ij}(q)}\right)$ of hesitant fuzzy linguistic element $h_S^{\nu_{ij}(q)}$ was calculated by using Equation (9). Thus, the hesitant fuzzy linguistic orthopair fuzzy decision matrix of the expert D_q $(q=1,2,3)$ was obtained, as shown in Table 6, Table 7 and Table 8 respectively. Taking $\left(h_S^{\mu_{11}(1)}, h_S^{\nu_{11}(1)}\right)$ as an example, $\hat{s}\left(h_S^{\mu_{11}(1)}\right)$ and $\hat{s}\left(h_S^{\nu_{11}(1)}\right)$ were calculated as follows:

$$\hat{s}\left(h_S^{\mu_{11}(1)}\right)=\frac{\theta_4+\theta_5}{2}=\frac{0.6901+0.8498}{2}=0.7700;$$

$$\hat{s}\left(h_S^{\nu_{11}(1)}\right)=\frac{\theta_1+\theta_2}{2}=\frac{0.1500+0.3097}{2}=0.2299.$$

Therefore, there was $\left(\hat{s}\left(h_S^{\mu_{11}(1)}\right), \hat{s}\left(h_S^{\nu_{11}(1)}\right)\right)=(0.7700, 0.2299)$.

Table 6. Hesitant fuzzy linguistic orthopair fuzzy decision matrix of the expert D_1.

C_j	A_1	A_2	A_3	A_4
C_1	(0.7700, 0.2299)	(0.3199, 0.7700)	(0.0750, 0.8466)	(0.2299, 0.6800)
C_2	(0.4049, 0.8466)	(0.4049, 0.7700)	(0.7700, 0.1532)	(0.7700, 0.2299)
C_3	(0.5951, 0.2299)	(0.4166, 0.3199)	(0.2299, 0.8466)	(0.5000, 0.4999)
C_4	(0.8466, 0.2299)	(0.9249, 0.1532)	(0.1532, 0.7600)	(0.0750, 0.9249)
C_5	(0.9249, 0.0750)	(0.5000, 0.4999)	(0.9249, 0.1532)	(0.0000, 0.9249)

Table 7. Hesitant fuzzy linguistic orthopair fuzzy decision matrix for expert D_2.

C_j	A_1	A_2	A_3	A_4
C_1	(0.8466, 0.1532)	(0.4049, 0.7700)	(0.2299, 0.7700)	(0.3097, 0.7700)
C_2	(0.4049, 0.8466)	(0.3199, 0.7700)	(0.7700, 0.2299)	(0.8466, 0.0750)
C_3	(0.4999, 0.6800)	(0.6800, 0.3199)	(0.2399, 0.6901)	(0.4049, 0.5951)
C_4	(0.8466, 0.0750)	(0.7700, 0.1532)	(0.1532, 0.9249)	(0.0750, 0.8498)
C_5	(0.9249, 0.0750)	(0.5951, 0.3097)	(0.8466, 0.1532)	(0.0000, 0.9249)

Table 8. Hesitant fuzzy linguistic orthopair fuzzy decision matrix for expert D_3.

C_j	A_1	A_2	A_3	A_4
C_1	(0.8466, 0.1532)	(0.4049, 0.7700)	(0.1532, 0.9249)	(0.2299, 0.7700)
C_2	(0.4049, 0.7700)	(0.2399, 0.7700)	(0.7700, 0.0750)	(0.8498, 0.0750)
C_3	(0.5951, 0.7700)	(0.3300, 0.4049)	(0.0750, 0.9249)	(0.4049, 0.7700)
C_4	(0.5951, 0.4049)	(0.6800, 0.2299)	(0.2299, 0.7700)	(0.1532, 0.9249)
C_5	(0.8498, 0.0750)	(0.5951, 0.3097)	(0.8466, 0.1532)	(0.0750, 0.9249)

Step 5: Established the hesitant fuzzy linguistic Pythagorean fuzzy decision matrix of the expert D_q.

According to step 5 in Section 4.2, based on Table 6, $\left(\hat{s}\left(h_S^{\mu_{ij}(1)}\right)\right)^2+\left(\hat{s}\left(h_S^{\nu_{ij}(1)}\right)\right)^2$ ($i=1,2,3,4$; $j=1,2,\cdots,5$) can be calculated. The calculation results showed that:

$$\max_{ij}\left(\hat{s}\left(h_S^{\mu_{ij}(1)}\right)\right)^2+\left(\hat{s}\left(h_S^{\nu_{ij}(1)}\right)\right)^2=0.4049^2+0.8466^2=0.8807<1.$$

Thus, it can be concluded that Table 6 was also the hesitant fuzzy linguistic Pythagorean fuzzy decision matrix of the expert D_1.

Based on Table 7, $\left(\hat{s}\left(h_S^{\mu_{ij}(2)}\right)\right)^2+\left(\hat{s}\left(h_S^{\nu_{ij}(2)}\right)\right)^2$ ($i=1,2,3,4$; $j=1,2,\cdots,5$) can be calculated. The calculation results showed that:

$$\max_{ij}\left(\hat{s}\left(h_S^{\mu_{ij}(2)}\right)\right)^2+\left(\hat{s}\left(h_S^{\nu_{ij}(2)}\right)\right)^2=0.1532^2+0.9249^2=0.8789<1.$$

Thus, it can be concluded that Table 7 was also the hesitant fuzzy linguistic Pythagorean fuzzy decision matrix of the D_2.

Based on Table 8, $\left(\hat{s}\left(h_S^{\mu_{ij}(3)}\right)\right)^2+\left(\hat{s}\left(h_S^{\nu_{ij}(3)}\right)\right)^2$ ($i=1,2,3,4$; $j=1,2,\cdots,5$) can be calculated. The calculation results showed that:

$$\max_{ij}\left(\hat{s}\left(h_S^{\mu_{ij}(3)}\right)\right)^2+\left(\hat{s}\left(h_S^{\nu_{ij}(3)}\right)\right)^2=0.5951^2+0.7700^2=0.9470<1.$$

Thus, it can be concluded that Table 8 was also the hesitant fuzzy linguistic Pythagorean fuzzy decision matrix of the D_3.

At this point, the hesitant fuzzy linguistic orthopair fuzzy number $\left(\hat{s}\left(h_S^{\mu_{ij}(q)}\right),\hat{s}\left(h_S^{\nu_{ij}(q)}\right)\right)$ was also the hesitant fuzzy linguistic Pythagorean fuzzy number $\left(\hat{s}\left(h_S^{\mu_{P,ij}(q)}\right),\hat{s}\left(h_S^{\nu_{P,ij}(q)}\right)\right)$.

Step 6: Calculated the hesitant fuzzy linguistic Pythagorean fuzzy decision matrix aggregating three experts' opinions.

According to the data in Table 6, Table 7 and Table 8, three hesitant fuzzy linguistic Pythagorean fuzzy numbers $\left(\hat{s}\left(h_S^{\mu_{P,ij}(q)}\right),\hat{s}\left(h_S^{\nu_{P,ij}(q)}\right)\right)$ ($q=1,2,3$) were aggregated by using Equation (10). Thus, the hesitant fuzzy linguistic Pythagorean fuzzy decision matrix aggregating three experts' opinions can be obtained, as shown in Table 9. Taking $\left(\hat{s}\left(h_S^{\mu_{P,11}(q)}\right),\hat{s}\left(h_S^{\nu_{P,11}(q)}\right)\right)$ as an example, $\hat{s}\left(h_S^{\mu_{P,11}}\right)$ and $\hat{s}\left(h_S^{\nu_{P,11}}\right)$ were calculated as follows:

$$\hat{s}\left(h_S^{\mu_{P,11}}\right)=0.7700^{0.3976}\times0.8466^{0.3012}\times0.8466^{0.3012}=0.8153;$$

$$\hat{s}\left(h_S^{\nu_{P,11}}\right)=\sqrt{1-\left(1-0.2299^2\right)^{0.3976}\times\left(1-0.1532^2\right)^{0.3012}\times\left(1-0.1532^2\right)^{0.3012}}=0.1878.$$

Therefore, there was $\left(\hat{s}\left(h_S^{\mu_{P,11}}\right),\hat{s}\left(h_S^{\nu_{P,11}}\right)\right)=\left(0.8153,0.1878\right)$.

Table 9. Hesitant fuzzy linguistic Pythagorean fuzzy decision matrix aggregating three experts' opinions.

C_j	A_1	A_2	A_3	A_4
C_1	(0.8153, 0.1878)	(0.3687, 0.7700)	(0.1303, 0.8614)	(0.2515, 0.7385)
C_2	(0.4049, 0.8271)	(0.3221, 0.7700)	(0.7700, 0.1647)	(0.8162, 0.1571)
C_3	(0.5647, 0.6170)	(0.4501, 0.3484)	(0.1662, 0.8495)	(0.4403, 0.6352)
C_4	(0.7613, 0.2726)	(0.7978, 0.1800)	(0.1731, 0.8352)	(0.0930, 0.9077)
C_5	(0.9016, 0.0750)	(0.5553, 0.4008)	(0.8769, 0.1532)	(0.0000, 0.9249)

Step 7: Determined the hesitant fuzzy linguistic Pythagorean fuzzy positive ideal solution S^+ and the hesitant fuzzy linguistic Pythagorean fuzzy negative ideal solution S^-.

According to the data in **Table 9**, we calculated the score of $\left(\hat{s}\left(h_S^{\mu_{P,ij}}\right), \hat{s}\left(h_S^{\nu_{P,ij}}\right)\right)$ [50]: $\hat{S}\left(P_{\hat{s}}^{ij}\right) = \left(\hat{s}\left(h_S^{\mu_{P,ij}}\right)\right)^2 - \left(\hat{s}\left(h_S^{\nu_{P,ij}}\right)\right)^2$. The results of the calculation were shown in **Table 10**. Taking $\left(\hat{s}\left(h_S^{\mu_{P,11}}\right), \hat{s}\left(h_S^{\nu_{P,11}}\right)\right)$ as an example, the score of $\left(\hat{s}\left(h_S^{\mu_{P,11}}\right), \hat{s}\left(h_S^{\nu_{P,11}}\right)\right)$ was calculated as follows:

$$\hat{S}\left(P_{\hat{s}}^{11}\right) = \left(\hat{s}\left(h_S^{\mu_{P,11}}\right)\right)^2 - \left(\hat{s}\left(h_S^{\nu_{P,11}}\right)\right)^2 = 0.8153^2 - 0.1878^2 = 0.6294.$$

Table 10. The scores of $\left(\hat{s}\left(h_S^{\mu_{P,ij}}\right), \hat{s}\left(h_S^{\nu_{P,ij}}\right)\right)$.

C_j	A_1	A_2	A_3	A_4
C_1	0.6294	−0.4570	−0.7250	−0.4821
C_2	−0.5202	−0.4892	0.5658	0.6415
C_3	−0.0618	0.0812	−0.6940	−0.2096
C_4	0.5053	0.6041	−0.6676	−0.8153
C_5	0.8073	0.1477	0.7455	−0.8554

Based on the data in **Table 10**, according to Equation (11), the hesitant fuzzy linguistic Pythagorean fuzzy positive ideal solution was S^+ = {(0.8153, 0.1878), (0.4049, 0.8271), (0.4501, 0.3484), (0.7978, 0.1800), (0.9016, 0.0750)}. According to Equation (12), the hesitant fuzzy linguistic Pythagorean fuzzy negative ideal solution was S^- = {(0.1303, 0.8614), (0.8162, 0.1571), (0.1662, 0.8495), (0.0930, 0.9077), (0.0000, 0.9249)}.

Step 8: Calculated the distance $D\left(A_i, S^+\right)$ between the alternative A_i and the hesitant fuzzy linguistic Pythagorean fuzzy positive ideal solution S^+, and the distance $D\left(A_i, S^-\right)$ between the alternative A_i and the hesitant fuzzy linguistic Pythagorean fuzzy negative ideal solution S^-.

From Equation (13), the distance $D\left(A_i, S^+\right)$ between the alternative A_i and the hesitant fuzzy linguistic Pythagorean fuzzy positive ideal solution S^+ was calculated. The results were shown in **Table 11**. From Equation (14), the distance between the alternative A_i and the hesitant fuzzy linguistic Pythagorean fuzzy negative ideal solution S^- was calculated. The results were shown in Table 11.

Step 9: Calculated the relative closeness $RC(A_i)$ of the alternative A_i to the hesitant fuzzy linguistic Pythagorean fuzzy positive ideal solution S^+ and the ranking of alternatives.

From Equation (15), the relative closeness $RC(A_i)$ of the alternative A_i to the hesitant fuzzy linguistic Pythagorean fuzzy positive ideal solution S^+ was calculated, and the ranking of the alternatives was then obtained. The decision results were shown in Table 11.

Table 11. Decision results.

Alternatives	$D(A_i, S^+)$	$D(A_i, S^-)$	$RC(A_i)$	Ranking
A_1	0.0549	0.6456	0.9217	1
A_2	0.2411	0.5035	0.6762	2
A_3	0.5124	0.1916	0.2722	3
A_4	0.6231	0.0782	0.1116	4

As shown in Table 11, according to the size of relative closeness $RC(A_i)$, the ranking of four alternatives is $A_1 \succ A_2 \succ A_3 \succ A_4$. This indicates that the credit status of alternative A_1 is the best and the credit status of alternative A_4 is the worst.

5.3. Comparative Analysis

To illustrate the feasibility and effectiveness of the TOPSIS method for MAGDM based on HFLPFS proposed in this paper (hereinafter referred to as Method 1), the decision results shown in Table 11 were compared with the decision results of the TOPSIS method for MAGDM based on linguistic Pythagorean fuzzy set (hereinafter referred to as Method 2). When other conditions remain unchanged, the linguistic Pythagorean fuzzy decision matrix aggregating three experts' opinions obtained by Method 2 was shown in Table 12 (Note: In this paper, linguistic Pythagorean fuzzy set is regarded as a special case of hesitant fuzzy linguistic Pythagorean fuzzy set). The decision results obtained by Method 2 were shown in Table 13 (Note: Due to space limitations, the calculation process is available upon request).

Table 12. Linguistic Pythagorean fuzzy decision matrix aggregating three experts' opinions.

C_j	A_1	A_2	A_3	A_4
C_1	(0.7776, 0.1826)	(0.3455, 0.7394)	(0.1342, 0.8355)	(0.2495, 0.7009)
C_2	(0.3935, 0.7874)	(0.3167, 0.7394)	(0.7392, 0.1634)	(0.7776, 0.1538)
C_3	(0.5375, 0.5953)	(0.3839, 0.3361)	(0.1630, 0.8258)	(0.4319, 0.6049)
C_4	(0.7296, 0.2689)	(0.7680, 0.1730)	(0.1726, 0.8066)	(0.0861, 0.8836)
C_5	(0.8736, 0.0772)	(0.5279, 0.3937)	(0.8448, 0.1442)	(0.0000, 0.9029)

Table 13. Decision results (Method 2).

Alternatives	$D\left(A_i,S^+\right)$	$D\left(A_i,S^-\right)$	$RC\left(A_i\right)$	Ranking
A_1	0.0553	0.7053	0.9273	1
A_2	0.2230	0.5910	0.7261	2
A_3	0.4773	0.2279	0.3232	3
A_4	0.5790	0.1128	0.1630	4

As shown in Table 13, according to the size of relative closeness $RC\left(A_i\right)$, the ranking of four alternatives is $A_1 \succ A_2 \succ A_3 \succ A_4$. It can be seen that the ranking result of alternatives obtained by Method 1 is completely consistent with the ranking result of alternatives obtained by Method 2, indicating the feasibility of Method 1.

We further investigated the discrimination of Method 1 for alternatives, as well as Method 2. Let the ranking of m alternatives A_i ($i=1,2,\cdots,m$) be $A_{(1)} \succ A_{(2)} \succ \cdots \succ A_{(m)}$, where (i) is the subscript of the alternatives ranked by $RC\left(A_{(1)}\right) > RC\left(A_{(2)}\right) > \cdots > RC\left(A_{(m)}\right)$. According to the discrimination algorithm given in [55], the discrimination calculation formula of Method 1 (or Method 2) for alternatives is as follows:

$$\rho=\sum_{i=2}^{m}\frac{RC\left(A_{(1)}\right)-RC\left(A_{(i)}\right)}{RC\left(A_{(1)}\right)}+\sum_{i=3}^{m}\frac{RC\left(A_{(2)}\right)-RC\left(A_{(i)}\right)}{RC\left(A_{(2)}\right)}+\cdots+\frac{RC\left(A_{(m-1)}\right)-RC\left(A_{(i)}\right)}{RC\left(A_{(m-1)}\right)} \quad (16)$$

From Equation (16), the discrimination of Method 1 for alternatives was calculated as

$$\begin{aligned}\rho_1&=\frac{0.9217-0.6762}{0.9217}+\frac{0.9217-0.2722}{0.9217}+\frac{0.9217-0.1116}{0.9217}\\&\quad+\frac{0.6762-0.2722}{0.6762}+\frac{0.6762-0.1116}{0.6762}+\frac{0.2722-0.1116}{0.2722}\\&=3.8724\end{aligned}$$

Similarly, the discrimination of Method 2 for alternatives was calculated as

$$\begin{aligned}\rho_2&=\frac{0.9273-0.7261}{0.9273}+\frac{0.9273-0.3232}{0.9273}+\frac{0.9273-0.1630}{0.9273}\\&\quad+\frac{0.7261-0.3232}{0.7261}+\frac{0.7261-0.1630}{0.7261}+\frac{0.3232-0.1630}{0.3232}\\&=3.5188\end{aligned}$$

It can be seen that although the ranking results of the alternatives obtained by Method 1 and Method 2 are completely consistent, the discrimination of Method 1 for alternatives is higher than that of Method 2, which means that the decision results obtained by Method 1 are better than those by Method 2, indicating the effectiveness of Method 1.

6. Discussion

According to the comparative analysis results in Section 5.3, it can be seen that in the case where the two ranking results of the alternatives were completely consistent, the discrimination of Method 1 for alternatives was 3.8724, which was higher than 3.5188 of Method 2. This is because, from the calculation of the data in Table 11 and Table 13, the standard deviation of the relative closeness $RC(A_i)$ obtained by Method 1 is 0.3704, which is higher than 0.3529 of Method 2. According to Equation (15), the specific reason is that the standard deviation of $D(A_i,S^+)/D(A_i,S^-)$ obtained by Method 1 is 3.6278, which is higher than 2.3184 of Method 2. This transmission mechanism is as follows:

$$\sigma_{D(A_i,S^+)/D(A_i,S^-)}\uparrow \Rightarrow \sigma_{RC(A_i)}\uparrow \Rightarrow \rho\uparrow$$

where $\sigma_{D(A_i,S^+)/D(A_i,S^-)}$ and $\sigma_{RC(A_i)}$ represents the standard deviation of $D(A_i,S^+)/D(A_i,S^-)$ and $RC(A_i)$, respectively, ρ represents the discrimination.

Obviously, the standard deviation of $D(A_i,S^+)/D(A_i,S^-)$ is closely related to the standard deviations of $D(A_i,S^+)$ and $D(A_i,S^-)$. Furthermore, according to Equation (13) and Equation (14), the standard deviations of $D(A_i,S^+)$ and $D(A_i,S^-)$ obtained by Method 1 are closely related to the elements in Table 9; similarly, the standard deviations of $D(A_i,S^+)$ and $D(A_i,S^-)$ obtained by Method 2 are closely related to the elements in Table 12.

In Method 1, the basis of Table 9 is the linguistic expression initial decision matrix of the expert D_q ($q=1,2,3$). This matrix is derived from, under the linguistic term set *S*, the expert D_q employs the linguistic expressions $ll \in S_{ll}$ generated by the context-free grammar G_H to evaluate the membership degree μ_{ij} and non-membership degree v_{ij} of alternative A_i with respect to attribute C_j. However, in Method 2, the basis of Table 12 is the linguistic term initial decision matrix of the expert D_q ($q=1,2,3$). This matrix is derived from the expert D_q employs the linguistic term in *S* to evaluate the membership degree μ_{ij} and non-membership degree v_{ij} of alternative A_i with respect to attribute C_j.

Method 2 has 7 linguistic terms (see Step 1 in Section 5.2), and Method 1 has 12 linguistic expressions (see Definition 4). It can be seen that compared with Method 2, Method 1 has a higher richness of linguistic elicitation. This may be reflected in: compared with the elements in Table 12, the elements in Table 9 are generally different in the standard deviations of membership degree, non-membership degree and indeterminacy degree. According to the calculation of the data in Table 9 and Table 12, the standard deviations of membership, non-membership and indeterminacy degree of all elements in Table 9 are respectively 0.2929, 0.3117 and 0.1103, which are respectively higher than 0.2804, 0.3002 and 0.1022 in Table 12.

7. Conclusions

The marginal contributions of this paper may be as follows: 1) Utilizing the linguistic scale function, the definition of the hesitant fuzzy linguistic element normalized score function (HFLENSF) was proposed, and the mapping from hesitant fuzzy linguistic term set (HFLTS) to [0, 1] interval was realized. Based on the HFLENSF, the definition of hesitant fuzzy linguistic Pythagorean fuzzy set (HFLPFS) was then proposed. Thus, the hesitant fuzzy linguistic term set was reasonably introduced into the Pythagorean fuzzy set. 2) The HFLPFS was applied to MAGDM, and a TOPSIS method for MAGDM based on HFLPFS was constructed. 3) The TOPSIS method for MAGDM based on HFLPFS was applied to the credit evaluation of the listed companies in strategic emerging industries in China. The example analysis results showed the feasibility and effectiveness of the proposed method, and proved the applicability of HFLPFSs in MAGDM.

The advantages of this paper may be as follows: 1) Compared with [1]-[31], this paper introduces hesitant fuzzy linguistic term set into Pythagorean fuzzy set, using hesitant fuzzy linguistic term set to describe the membership degree and non-membership degree of the element $x_i \in X$ to Pythagorean fuzzy set, which improves the richness of linguistic elicitation, which is more reasonable. 2) Compared with [34], this paper theoretically defines the ranges of membership degree and non-membership degree characterized by hesitant fuzzy linguistic elements, which is more scientific. 3) Compared with [38], the ranges of membership degree and non-membership degree characterized by hesitant fuzzy linguistic elements defined by this paper are consistent with the ranges of membership degree and non-membership degree defined by orthopair fuzzy sets, which is more reasonable. 4) Compared with [34]-[41], this paper extends the application field of hesitant fuzzy linguistic term set from intuitionistic fuzzy sets to Pythagorean fuzzy sets, which is more generalized.

The academic value of this paper may be reflected as follows: 1) The definition of hesitant fuzzy linguistic Pythagorean fuzzy set (HFLPFS) proposed in this paper provides a new method and idea for exploring the determination method of membership degree and non-membership degree with higher richness of linguistic elicitation, which has high theoretical value. 2) The TOPSIS method for MAGDM based on HFLPFS proposed in this paper can be widely applied to MAGDM problems, which has high practical application value and broad application prospects. In summary, this study enriches the theoretical framework of Pythagorean fuzzy sets, and expands the applicability and methodology of Pythagorean fuzzy sets in MAGDM.

In this paper, only three experts were set in the application example analysis, and the number was small. In the future, the number of experts will be increased to improve the stability of MAGDM and the persuasiveness of the application example analysis results. At the same time, this paper used the cyclic mutual evaluation method to determine the weights of experts in the application example analysis, which had certain limitations. The future application example analysis will consider introducing more advanced methods [56] [57] to determine the weights of experts to improve the rationality of the weights of experts. In addition, for the sake of simplicity, this paper only selected five indicators to evaluate the credit status of sample enterprises in the application example analysis, which led to incomplete credit information. The future application example analysis will consider adding enterprise credit evaluation indicators to improve the accuracy of enterprise credit evaluation. In addition, Method 1 should be compared with other methods besides Method 2, such as two-tuple linguistic representation [29] [30], linguistic hesitant fuzzy sets [31], etc.

"Data Availability" Statements

The datasets used and/or analyzed during the current study are available from the corresponding author on reasonable request.

Funding

This research was funded by the Regional Project of National Natural Science Foundation of China, grant number 71861003.

Availability of Data and Material

Most of the data generated or analyzed during this study are included in the manuscript, and the rest data are promptly available to readers without undue qualifications.

Code Availability

Not applicable.

Author Contributions

Conceptualization, Mu Zhang; methodology, Mu Zhang; validation, Mu Zhang and Yan Li; formal analysis, Mu Zhang and Yan Li; investigation, Yan Li; resources, Mu Zhang; data curation, Mu Zhang and Yan Li; writing—original draft preparation, Mu Zhang and Yan Li; writing—review and editing, Mu Zhang and Yan Li; supervision, Mu Zhang; project administration, Mu Zhang; funding acquisition, Mu Zhang. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.

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Appendix A

Table A1. Linguistic expression initial decision matrix of the expert D_1.

C_j	A_1	A_2	A_3	A_4
C_1	{between high and very high, between very low and low}	{between very low and medium, between high and very high}	{less than low, at least high}	{between very low and low, between medium and very high}
C_2	{between low and medium, at least high}	{between low and medium, between high and very high}	{between high and very high, at most low}	{between high and very high, between very low and low}
C_3	{between medium and high, between very low and low}	{at most very high, between very low and medium}	{between very low and low, at least high}	{medium, between low and high}
C_4	{between high and extremely high, between very low and low}	{at least very high, between extremely low and low}	{between extremely low and low, at least medium}	{at most very low, at least very high}
C_5	{between very high and extremely high, between extremely low and very low}	{medium, between low and high}	{at least very high, at most low}	{extremely low, at least very high}

Table A2. Linguistic expression initial decision matrix of the expert D_2.

C_j	A_1	A_2	A_3	A_4
C_1	{between high and very high, at most low}	{between low and medium, between high and very high}	{between very low and low, between high and very high}	{low, between high and very high}
C_2	{between low and medium, at least high}	{between very low and medium, between high and very high}	{between high and very high, between very low and low}	{between high and very high, at most very low}
C_3	{between low and high, between medium and very high}	{between medium and very high, between very low and medium}	{between extremely low and medium, high}	{between low and medium, between medium and high}
C_4	{between high and very high, at most very low}	{between high and very high, at most low}	{between extremely low and low, at least very high}	{between extremely low and very low, very high}
C_5	{between very high and extremely high, between extremely low and very low}	{between medium and high, low}	{at least very high, at most low}	{extremely low, at least very high}

Table A3. Linguistic expression initial decision matrix of the expert D_3.

C_j	A_1	A_2	A_3	A_4
C_1	{between high and very high, at most low}	{between low and medium, between high and very high}	{between extremely low and low, at least very high}	{between very low and low, between high and very high}
C_2	{between low and medium, between high and very high}	{at most medium, between high and very high}	{between high and very high, between very low and low}	{very high, at most very low}
C_3	{between medium and high, between high and very high}	{at most high, between low and medium}	{at most very low, at least very high}	{between low and medium, between high and very high}
C_4	{between medium and high, between low and medium}	{between medium and very high, between very low and low}	{between very low and low, between high and very high}	{between extremely low and low, at least very high}
C_5	{very high, between extremely low and very low}	{between medium and high, low}	{between high and extremely high, at most low}	{between extremely low and very low, at least very high}