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Estimation of Value at Risk of Oil Prices Using Normal, Stable, and Generalized Hyperbolic Distributions

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Abstract

This study applies the variance-covariance technique to estimate the Value at Risk (VaR) of OPEC crude oil prices. To capture the variability, heavy tails, and uncertainty in the dataset, three distributions were utilized within an ARIMA-GARCH framework: Normal, α-Stable, and Generalized Hyperbolic Distribution (GHD). The efficacy of the estimated VaR was evaluated using Kupiec test, Christoffersen test, and Lopez's loss function. At 95% confidence level: ARIMA-GARCH-Normal and Monte Carlo Simulation produced more precise VaR estimates according to the Kupiec test, while only ARIMA-GARCH-Normal produced accurate estimates according to the Christoffersen test. Based on Lopez's loss function, the ranking from best to worst is: Normal, Monte Carlo, Stable, GHD, Historical Simulation. At 99% confidence level: ARIMA-GARCH-Normal, ARIMA-GARCH-Stable, and ARIMA-GARCH-GHD are statistically accurate under both the Kupiec and Christoffersen tests. Based on Lopez's loss function, the ranking from best to worst is: Stable, Normal, GHD, Monte Carlo, Historical Simulation. The findings indicate that ARIMA-GARCH-Normal is the most consistently accurate model across confidence levels. It is recommended that risk analysts adopt Normal and Stable distributions within an ARIMA-GARCH framework for estimating VaR of OPEC oil prices, as these distributions provide the most accurate and stable estimations across the backtesting criteria.

Keywords

Value-at-Risk, Normal Distribution, Stable Distribution, Generalized

Hyperbolic Distribution, Lopez Loss Functions

1. Introduction

Crude oil serves as a primary source of energy; it accounts for more than thirty-three percent of global energy usage [1]. Similar to other primary commodities, oil plays a crucial role in industrial output and as such oil price fluctuations affect price levels, economic processes and stock market [2] [3]. The critical role oil plays in the boosting of most economies worldwide became evident in the oil crisis in the 1970's and the year 2000 [4]. Fluctuations in oil prices affect not only market participants but also government and society as a whole [5]. According to [6] while changes in oil price fluctuations affects economic activities significantly and oil prices are unresponsive to changes in economic activities. Oil price shocks have an impact economic activity through different supply and demand channels [7].

Consumers, producers and government are very concerned about oil prices and their excessive volatility and there is a growing academic research focus on this crucial economic commodity in risk mitigation. A lot of factors contributes to fluctuations in oil prices which includes policies of the Organization of Petroleum Exporting Countries (OPEC), militarily conflicts, natural disasters and supply and demand imbalances in international market [8]. Due to the uncertainty and volatility in the oil market environment it is vital to apply effective tools for managing the risk associated with fluctuations in oil prices [9].

Value at Risk (VaR) is a key metric utilized to quantify price movement risks [10] [11]. It represents market risk using probability distribution and quantifies it with a single value [11]. Thus, VaR has become a key tool used in the finance for risk quantification [12]. A study was conducted by [13] to estimate the risk associated with the stock returns of Indian stock market from 1996 to 2014. Based on the findings of the study, there exist a positive risk premium linked with VaR in the Indian Stock market for the period spanning 2001-2008. [14] worked on VaR estimation of metals utilizing Long Memory GARCH models.

VaR can be applied in oil market for risk quantification and various authors have done notable research in this field [15]. [16] applied VaR to estimate oil price variation for designing risk management strategies [17]. [18] conducted a study to construct a probabilistic model that applies VaR to estimate the risk of production for oil and gas projects. [19] utilized GARCH-type model, EVT and Vine copular to measure VaR and Expected Shortfalls of portfolios consisting of four crude oil assets. [20] worked on estimating and forecasting portfolio VaR with Extreme Value Theorem. It is important to understand that risk quantification metrics, such as VaR depend heavily on underlying distributional assumptions [14]. Thus, the need for a distribution that can cover the entire financial data which gives a big advantage to both risk managers and investors.

In recent times, more complex tools for assessing oil price risk have been developed due to financial crisis and rise in market volatility [21]. VaR is a well-known risk metric used to quantify risk in the oil market [22] [23]. It is used in the estimation of tails of empirical distributions of financial data. Historically, commodity prices exhibit highest volatility pr among international prices [24]. Commodity price volatility is usually greater than that of exchange and interest rate volatilities. Although financial returns typically follows a normal distribution, it is widely known that financial data also exhibits features such as non-zero skewness, volatility, absence of autocorrelation, and long-range dependence [25] [26]. However, limited research exists on modeling alternative asset volatilities using non-normal distributions A class of continuous distributions referred to as Generalized Hyperbolic Distributions was introduced by [27] to capture excess kurtosis where the probability density function's logarithm is a hyperbola. Relative to some distributions like the normal and Student-t distributions, these distributions fit financial returns more effectively. Stable distributions are also a category of four parameter models which are able to account for skewness and heavy tail of financial data used in VaR estimations. Authors such as [28], for instance, demonstrated that GHDs offered greatest fit to model high-frequency data using daily prices of the 30 DAX spanning for three years. [29] and [30] conducted related studies. GHDs were also used by [31] to calculate the appropriate VaR values using the South African Mining Index. The generalized skew t-type distributions were proposed for modeling of data by a number of researchers.

However, a very few studies have been conducted on VaR estimation of oil prices and using the generalized hyperbolic distribution methods. This study aims to estimate VaR of oil prices via econometric approach using ARMA-GARCH-Normal, ARIMA-GARCH-Stable and ARIMA-GARCH-GHDs on global oil prices. To be specific this study extends a paper by [8] by utilizing Kupiec Likelihood (KLT) Test and the Christofferson Conditional Coverage Test (CCT) for VaR in backtesting and compares the efficacy of both stable and hyperbolic distributions in estimating VaR of OPEC crude oil prices. The proposed approach captures some key stylized facts about oil prices including clustering, asymmetry, heavy tails, leverage effect and volatility. The proposed models exhibit notable strengths in capturing conditional heteroscedasticity within the returns process through the GARCH approach, while concurrently accommodating heavy-tailed behavior via stable distributions and Generalized Hyperbolic Distributions (GHDs).

2. Materials and Methods

This study estimates and compares Value at Risk (VaR) for OPEC oil prices using five approaches. The three core models are based on the variance-covariance technique with ARIMA-GARCH filtered residuals under three distributional assumptions: Normal, α-Stable, and Generalized Hyperbolic Distribution (GHD). For benchmarking purposes, two additional non-parametric/simulation approaches are also evaluated: Monte Carlo Simulation and Historical Simulation. All five models are backtested using Kupiec test, Christoffersen test, and Lopez's loss function.

2.1. Value at Risk

VaR is an indicator which provides the maximum expected loss of an investment, or a trading position given a confidence level β and a holding period $\boldsymbol{q}$. Let $y_t, t = 0,1,\cdots,n$ be the series of returns or profit and loss (P/L) of an investment. For a long position, VaR is the α-quantile of the distribution of y mathematically expressed in Equation (1)

$$VaR_\alpha = F^{-1}(\alpha) \tag{1}$$

where $\alpha = 1-\beta$ and F are the significance level of the cumulative distribution function. For short positions, the same formula is used with the distribution of $-y$. Three techniques are utilized for VaR, namely Historical, Monte Carlo simulation method and Parametric or Variance Covariance method. This research employed the variance covariance technique.

Risk Metrics

An investor can determine VaR of a portfolio of investment using the assumptions that the returns on any asset or portfolio are distributed normally over time. VaR is calculated by utilizing the variance-covariance technique and the volatility and correlation of the data set. The original mode of VaR estimation is mathematically expressed in Equation (2).

$$p_t = p_{t-1} + \mu_t + \sigma_t \epsilon_t \tag{2}$$

where p_t, μ_t, and σ_t are the log price of the asset, the conditional mean and variance and $\epsilon_t \sim N(0,1)$. In the particular Risk Metrics consider $\mu_t = 0$ therefore

$$p_t - p_{t-1} = r_t = \sigma_t \epsilon_t \tag{3}$$

Finally, it is considered that, in this case $r_t \sim N(0,\sigma_t)$ and VaR could be calculated as the variance σ_t times the quantile of normal distribution with zero mean and variance(Equation (4)),

$$\sigma_t : VaR_{t|t-1}(\alpha) = \sigma_t F^{-1}(\alpha) \tag{4}$$

where F is the $N(0,\sigma_t)$ CDF. In this research other distribution for ϵ_t is utilized and the conditional mean μ_t will be non-zero and forecasted by AR or ARMA process. Thus, VaR measure will be mathematically expressed in Equation (5):

$$VaR_{t|t-1}(\alpha) = \mu_t + \sigma_t F^{-1}(\alpha) \tag{5}$$

where F is the CDF of a given distribution.

2.2. Time Series Models

It must be clearly understood that trend and volatility models often feature in financial time series analysis. Classic mean models like moving average, ARIMA, and others typically capture the trend component. Thus, time series models were employed in this study; Time series analysis refers to techniques or procedures that dissect a series into discrete, explicable parts, enabling the detection of patterns, the establishment of estimates, and the generation of forecasts.

2.2.1. Stationary or Non-Stationary

Time series data could be stationary or non-stationary processes based on the characteristics it exhibits. Stationary time series has mean, variance, autocorrelation, invariant to time shifts non-stationary data exhibit regular trends like linear and quadratic ones. Homogenous series refers to a non-stationary time series which usually becomes stationary after differencing. Transforming time series to achieve stationarity is the answer and solution to having accurate findings and better forecasting. There are different models which can be used for the analysis; namely the autoregressive models, moving average models and a combination of autoregressive and moving averages to have autoregressive moving average (ARMA) and when differenced for a non-stationary time series data we have an ARIMA model. The various types of models employed in this study are explained below.

2.2.2. Auto Regressive (AR) Model

AR model entails a linear regression framework wherein the current value of a series is regressed on preceding (lagged) values. The model's order is defined by the value of p. Standard linear least squares procedures are one of the many strategies that are used to analyze autoregressive models. A well-known technique for modelling is the autoregressive model and it is denoted mathematically expressed in Equation (6).

$$Y_t = c + \sum_{i=1}^{p} \theta_i Y_{t-1} + \varepsilon_t \tag{6}$$

where $\theta_1, \cdots, \theta_p$ are the parameters, ε_t is the random variable and c is a constant.

2.2.3. Moving Average (MA) Model

Conceptually speaking, MA model entails regressing the current value of a series on past random shocks or errors. The error components in the moving average cannot be observed, making it more difficult than the autoregressive technique. Compared to autoregressive models, the interpretation of the moving average is less clear. The value of q is the order of the moving average method. MA model is represented mathematically by the equation below.

$$Y_t = \mu + \varepsilon_t + \sum_{t=1}^{q} \phi \varepsilon_{t-1} \tag{7}$$

where $\phi_1, \cdots, \phi_q$ are model parameters, μ represents the mean of the series and $\varepsilon_1, \cdots, \varepsilon_{t-1}$ are the white noise.

2.2.4. Autoregressive Moving Average Model of Order *p*, *q*; ARMA (*p*, *q*)

Given that $\{Y_t\}$ is an Autoregressive Moving Average process, ARMA (p, q) if $\{Y_t\}$ is stationary and then for every t, the autoregressive moving average model can be expressed in the Equation (8).

$$Y_t = \theta_1 Y_{t-1} + \theta_2 Y_{t-2} + \cdots + \theta_p Y_{t-p} + \varepsilon_t - \phi_1 \varepsilon_{t-1} - \phi_2 \varepsilon_{t-2} - \cdots - \phi_q \varepsilon_{t-q} \tag{8}$$

From the Equation (8), it can be concluded that ARMA comprises of the autoregressive and the moving average processes. The importance of the ARMA process is that the stationary time series is expressed with fewer parameters in the ARMA model than by a pure AR or MA process.

2.2.5. Autoregressive Integrated Moving Average Model of Order *p, d, q*

The ARIMA (p, d, q) model was introduced by Box and Jenkins in 1976. In these models, p, d and q represent the number of autocorrelation terms, the number of differencing components, and the number of moving average terms respectively. "I" in ARIMA represents the integration (differencing) needed to make non-stationary series stationary This model is applied in a situation where the data is not stationary. To fit a stationary model, the non-stationary source of variation is usually removed. Hence the ARMA model is differenced by replacing Y_t with $\Delta^d Y_t$. The model is termed "integrated" as it involves summing (integrating) the stationary model fitted to differenced data to a model the original non-stationary data. It is expressed in the form ARIMA (p, d, q).

p, d and q represents the lag observations in the model; referred to as the lag order, the number of times that the raw observations are differenced; termed as the degree of differencing and q the size of the moving average window; referred to as the order of the moving average respectively.

$$Y_t = \Delta^d Y_t = (1-B)^d Y_t \tag{9}$$

Equation (10) describes the general ARIMA model

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \varepsilon_t - \phi_1 \varepsilon_{t-1} - \phi_2 \varepsilon_{t-2} - \cdots - \phi_q \varepsilon_{t-q} \tag{10}$$

2.3. GARCH and IGARCH Models

Financial time series often exhibit stylized facts, making volatility modeling a key consideration. ARCH/GARCH modeling is an option worth considering for estimating values of the volatility. Furthermore, stylized facts are what cause most of the volatility in business and economic statistics GARCH models posit that the error term variance evolves according to an ARMA process. This statistical method is utilized to predict the volatility of financial asset returns. GARCH suits time series data with serially correlated error variance, following an ARMA pattern. It is applied to evaluate risk and expected returns for assets exhibiting clustered volatility patterns in their returns. Given the prevalent heteroskedasticity in time series data, it is imperative to employ a model that accommodates conditional heteroskedasticity. The GARCH model is employed in the study. The ARCH model accommodates long lags in conditional variance, while the GARCH model expands on this by allowing both long lags and a more flexible lag structure. The GARCH (p, q) model is described by Equation (11).

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-1}^2 + \sum_{i=1}^{p} \beta_i \sigma_{t-1}^2 \tag{11}$$

where p, σ_t^2 and q denotes the order of the GARCH terms, the order of the

ARCH terms ε_t^2.

The GARCH model is adept at capturing financial data features, notably clustering, volatility and leptokurtosis. A notable limitation of the GARCH model is its dependence on squared ε_t^2 values, which renders it sensitive to the magnitude of shocks but insensitive to their sign, thereby potentially failing to capture leverage effects, thereby potentially failing to capture leverage effects of returns. which exhibits persistence in variance, with current information remaining pertinent for forecasts across all time horizons. First, we consider the lag operator $L^k \alpha_t = \alpha_{t-k}$ and the lag polynomials $\alpha(L) = \alpha_1 L + \cdots + \alpha_q L^q$ and $\beta(L) = \beta_1 L + \cdots + \beta_p L^p$. Then the GARCH models are mathematically expressed as:

$$\sigma_t^2 = \alpha_0 + \alpha(L)\alpha_t^2 + \beta(L)\sigma_t^2 \tag{12}$$

and in the case when $\alpha(L) + \beta(L) = 1$ then the GARCH model may be represented as an ARMA (m, p) model where $m = \max(p, q)$,

$$(1 - \alpha(L) - \beta(L))\alpha_t^2 = \alpha_0 + (1 - \beta(L))v_t \tag{13}$$

where $v_t = \alpha_t^2 - \sigma_t^2$. From the above equation and considering the autoregressive polynomial $(1 - \alpha(L) - \beta(L))$, when this polynomial had $d > 0$ unit roots, and $m - d$ roots outside the circle, the GARCH model is integrated in variance of order d [IGARCH (p, d, q)]. Thus, the IGARCH model is the GARCH model where $\alpha(L) + \beta(L) = 1$.

Therefore, it can be deduced that the equation $\sigma_t^2 = \alpha\sigma_{t-1}^2 + (1-\alpha)r_{t-1}^2$ follow the model IGARCH (1, 1), which is the log returns assumed to fit the data.

The ARIMA-GARCH framework is employed to capture linear dependencies in both the conditional mean and variance, thereby ensuring a more comprehensive representation of the underlying data generating process. However, if volatility of data is constant, an ARIMA model is sufficient, as the GARCH component is tailored to capture instances of time-varying fluctuations. In modeling financial time series, an ARMA-GARCH framework is commonly employed to jointly capture the mean and volatility dynamics. The ARMA component addresses serial correlation in returns, while the GARCH component models address conditional heteroscedasticity, yielding a comprehensive representation of the underlying data generating process.

2.4. Monte Carlo Simulation

Monte Carlo Simulation (MCS) estimates VaR by generating a large number of possible future return scenarios. In this study, MCS is applied within the fitted ARIMA-GARCH framework. We simulate 10,000 standardized shocks from a Normal distribution and feed them into the estimated ARIMA-GARCH model to produce 10,000 one-day-ahead simulated returns. The 95% and 99% VaR are then obtained as the 5th and 1st percentiles of the simulated return distribution. MCS captures GARCH volatility dynamics without relying on a closed-form variance-covariance formula and was used as a simulation-based benchmark.

2.5. Historical Simulation

Historical Simulation (HS) is a non-parametric method that estimates VaR directly from past data without assuming any distribution. In this study, HS uses a rolling window of the most recent 500 daily log-returns. For each day, VaR at 95% and 99% is computed as the 5th and 1st percentiles of the returns in the window. As the window moves forward, the VaR estimate is updated. HS is included as a benchmark because it naturally incorporates skewness and fat tails present in OPEC oil returns, but it assumes that future returns will resemble the historical pattern.

2.6. Distributions

The assumption that the value-at-risk method for risk quantification depends highly on the distributions utilized will demand that some probability distributions been used, thus various distributions are employed to estimate value-at-risk.

2.6.1. Value at Risk Estimation under Normal Distribution

The normal distribution, also known as the Gaussian distribution, is a symmetric probability distribution in which observations are most densely concentrated around the mean and occur with decreasing frequency as distance from the mean increases. In graphical form it has a bell shape. This distribution is characterized by a mean of zero and a standard deviation of 1. Equation (14) defines the Probability Density Function (pdf)of a normal distribution,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{14}$$

where: y = value of the variable or data being examined, $f(x)$ is the pdf, μ is the mean and σ is the standard deviation.

2.6.2. Value at Risk Estimation under Levy Stable Distribution

The family of α-stable distributions, also referred to as stable distributions, constitutes a family of probability distributions characterized by their ability to accommodate asymmetry and heavy tails. In a seminal contribution, the French mathematician Paul Lévy (1920) delineated this class of distributions within the context of sums of independent, identically distributed terms. Stable distributions are characterized by their characteristic function (CF), as explicit analytical expressions for the probability density function (PDF) and cumulative distribution function (CDF) are generally unavailable. A random variable X is defined to have a stable distribution if there are parameters $0 < \alpha \le 2$, $-1 \le \beta \le 1$, and a real value μ such that its' characteristic function has the form

$$\phi(t,\alpha,\beta,\mu,\sigma) = \mathrm{e}^{it\mu - |ct|^{\alpha}(1 - i\beta\,\mathrm{sign}(t)\Phi(\alpha,t))} \tag{15}$$

2.6.3. Value at Risk Estimation under GHD

GHD, which was proposed by Barndorff-Nielsen in 1977 is a continuous probability distribution that is characterized as variance-mean mixes of generalized inverse Gaussian distributions. Because the logarithms of densities have a hyperbolic form whereas those of the normal distribution have a parabolic shape, the class is so named. GHDs are continuous distributions with five parameters that are excellent for modeling a range of data because they can accommodate asymmetry and data with heavy and semi-heavy tails. For the parameterization of univariate GHD. If X were a random variable that obeyed GHD, its probability density function (pdf) would be defined as $X \sim GH(\lambda, \alpha, \beta, \delta, \mu)$ where: μ, α, β are the location parameter, scale parameter and shape parameter respectively β determines the and λ influences the kurtosis and characterizes the classification of the GHD.

$$f_{GHD}(X) = \frac{\left(\alpha^2 - \beta^2\right)^{\frac{\gamma}{2}} K_{\gamma - \frac{1}{2}}\left(\alpha \sqrt{\delta^2 + (x-\mu)^2}\right) \exp\left(\beta (x-\mu)\right)}{\sqrt{2\pi} \alpha^{\lambda - \frac{1}{2}} \delta^{\lambda} K_{\lambda}\left(\delta \sqrt{\alpha^2 - \beta^2}\right) \left(\sqrt{\delta^2 + (x-\mu)^2}\right)^{\frac{1}{2} - \lambda}} \tag{16}$$

In the above expression, K_j is the modified Bessel function of the third kind of order j [32] and the parameters must fulfill the following conditions: $\delta \geq 0$, $|\beta| < \alpha$, if $\lambda > 0$, $\delta > 0$, $|\beta| < \alpha$, if $\lambda = 0$ and $\delta > 0$, $|\beta| \leq \alpha$, if $\lambda < 0$.

2.7. Backtesting Methods

Backtesting was employed to assess the risk predictive accuracy against the actuals. In this study three backtesting techniques were utilized to determine the accuracy of VaR estimates, namely Kupiec Likelihood Ratio (KLR) test, Christofferson Conditional Coverage (CCC) test and Lopez Magnitude Loss function (LML). The backtesting technique utilized in determining the accuracy of the VaR follows a chi-squared distribution.

2.7.1. Kupiec Likelihood Ratio Test

This test leverages the property that an adequate model should yield a proportion of VaR estimate violations that is near the specified probability level. The technique involves the calculation of the number of times x^{α} the observed returns fall below (for long positions) or above (for short positions) the VaR estimate at level α, *i.e.*, $r_t > VaR^{\alpha}$ or VaR^{α} and compare the corresponding failure rates to α. Under the null hypothesis that the expected proportion of violations equals α, the Kupiec statistics are mathematically expressed as

$$LR_{UC} = 2\ln\left(\left(\frac{x^{\alpha}}{N}\right)^{X^{\alpha}} \left(1 - \frac{x^{\alpha}}{N}\right)^{N - x^{\alpha}}\right) - 2\ln\left(\alpha^{x^{\alpha}} (1-\alpha)^{N - x^{\alpha}}\right) \tag{17}$$

The statistics follow an asymptotic chi-squared distribution with one degree of freedom.

2.7.2. Christoffersen Conditional Coverage Test

The Christoffersen test builds upon the Kupiec likelihood ratio test to account for violations of serial independence, thereby providing an indication of whether exceedances exhibit clustering. Clustering of exceedances implies a tendency for large portfolio losses to be followed by further large losses. This test is described in greater detail by [33]. The test statistic is expressed as

$$LR_{CC} = LR_{UC} + 2\ln\frac{\left[(1-\pi_0)^{\varphi_{00}}\,\pi_0\varphi_{01}\,(1-\pi_1)^{\varphi_{10}}\,\pi_1\varphi_{11}\right]}{\ln\left[(1-\pi)^{(\varphi_{00}+\varphi_{10})}\,\pi(\varphi_{01}+\varphi_{11})\right]} \tag{18}$$

where φ_{ij} is defined as the number of returns in state i while they have been in state j previously (state 1 indicates that VaR estimate is violated and state 0 indicates that VaR estimate is not violated) and π_i is defined as the probability of observing an exception given state i in the previous period. The statistic follows an asymptotic chi-squared distribution with two degrees of freedom.

2.7.3. Lopez Magnitude Loss Function

To compare different VaR models, the Lopez measure is used. It is a distance measure between the realized returns and the calculated VaR. The lower the Lopez measure the better the VaR measure. But this measure is only useful to compare different VaR and could not judge the accuracy of a given VaR.

$$L_t\left(x_t, VaR_t(\alpha)\right) = 1 + \begin{cases} 1 + \left[x_t - VaR_t(\alpha)\right]^2, & x_t \le VaR_t \\ 0, & x_t > VaR_t \end{cases} \tag{19}$$

$$L = \frac{1}{n}\sum_{t=1}^{n} L_t\left(x_t, VaR_t(\alpha)\right) \tag{20}$$

3. Data

Daily spot prices of OPEC crude oil, average, USD per barrel were obtained from the World Bank Commodity Price Data, for the period January 2, 1986, to February 28, 2020. The series represents the average of Dubai, Brent and WTI prices as reported by OPEC. The dataset consists of 8902 daily observations after adjustment for non-trading days. Non-trading days including weekends and holidays were forward-filled with the last available price, and log-returns were computed accordingly. The daily log return series r_t is computed as the first difference of logarithmic prices:. For day t, the daily log return r_t is expressed as

$$r_t = \ln P_t - \ln P_{t-1} \tag{21}$$

where P_t is the price at day t.

The plot of the daily oil prices from January 1986 to February 2020 are presented in **Figure 1** and it can be observed from **Figure 1** that the time series financial data shows increasing trends, but it is not stationary, thus exhibiting some characteristics like non-zero mean, and its variance is also not constant. Isolated extremes witnessed in the data are in July 1990 when the members of the OPEC initiated negotiations to curtail the supply of oil to the global market which caused: The July 2008 oil price spike marked the culmination of a decade-long energy crisis, driven by factors including Heightened demand from emerging economies, flat production levels, financial speculation, and escalating Middle East tensions., which collectively contributed to a steady increase in oil and gas prices. Also, the decline in oil prices from mid-2014 to early 2015 was primarily attributed to supply-side factors, which includes the surge in U.S. oil production, diminished geopolitical instability, and changes in OPEC's policy stance. In spring 2020, as covid spread Global lockdowns triggered a sharp decline in crude oil prices amid a collapse in demand. Russia-Ukraine conflict also affected the prices of oil in 2022.

The plot of the daily oil price returns from January 1986 to February 2020 is presented in **Figure 2**. When the log returns of the series are found as shown in

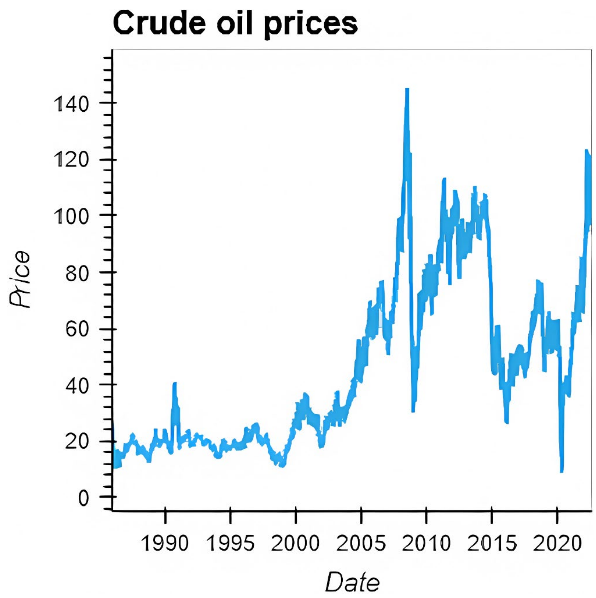

Figure 1. Time series plot of daily OPEC crude oil prices.

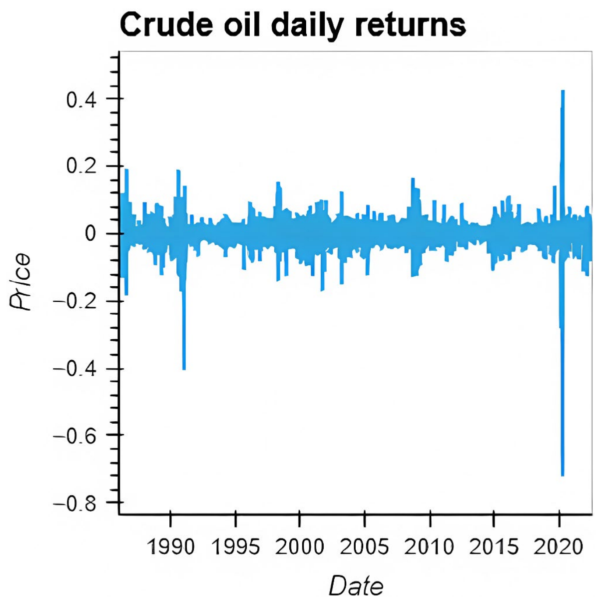

Figure 2. Plot of the daily OPEC oil price returns.

Figure 2, it was observed that the series was stationary, having a constant variance and a zero mean. Visual inspection of the plots reveals evidence of heteroskedasticity and volatility clustering across the return series.

3.1. Descriptive Statistics of the Data

Table 1 presents summary statistics for oil returns. It can be observed from Table 1 that; The oil returns exhibit a positive mean, suggesting a slight overall increase in returns over the examined period. The excess kurtosis value implies that these return series exhibit leptokurtic behavior. The returns exhibit tails substantially fatter than a normal distribution and is skewed towards the left with a negative value for skewness.

Table 1. Descriptive statistics of OPEC crude oil prices.

Variable	Minimum	Maximum	Mean	Standard Deviation	Kurtosis	Skewness
Statistics	−0.1765	0.0832	0.00035	0.0108	13.5349	−0.6291

3.2. Autocorrelation Plots

The ARIMA(p,d,q) model is identified using the standard Box-Jenkins methodology. First, the order of differencing d is determined by testing for stationarity of the log-return series using the Augmented Dickey-Fuller (ADF) test. The ADF test rejects the null hypothesis of a unit root at the 1% level, indicating that the return series is stationary in levels, thus $d = 0$. However, for consistency with standard financial modeling practice and to remove any potential trend in price levels, we model the log-prices and difference once, giving $d = 1$. Second, the autoregressive order p and moving average order q are identified by examining the Autocorrelation Function (ACF) and Partial Autocorrelation Function (PACF) of the stationary/differenced series. The PACF is used to identify p as the lag beyond which the PACF cuts off to zero. The ACF is used to identify q as the lag beyond which the ACF cuts off to zero. Based on the ACF and PACF plots of the first-differenced OPEC log-prices, the PACF shows no significant spikes beyond lag 0 and the ACF shows a significant spike at lag 1 followed by a cut-off. This pattern suggests an MA (1) process. Therefore, we select an ARIMA (0, 1, 1) model for the mean equation.

Figure 3 presents the ACF and PACF plots, which indicate that the series is non-stationary, necessitating further differencing. Although the log-return series is stationary, the ACF/PACF plots suggested an ARIMA (0, 1, 1) structure. The differencing term here captures first-order dependence in the mean and improves residual whiteness for the subsequent GARCH modeling of volatility.

Figure 4 shows the ACF and PACF plot of the differenced data. From Figure 4, it can be observed that the lags with highest spikes are 0 for the ACF plot and 1 for the PACF plot. Where the data was differenced only once, thus in conclusion the series follows ARIMA (0, 1, 1) model.

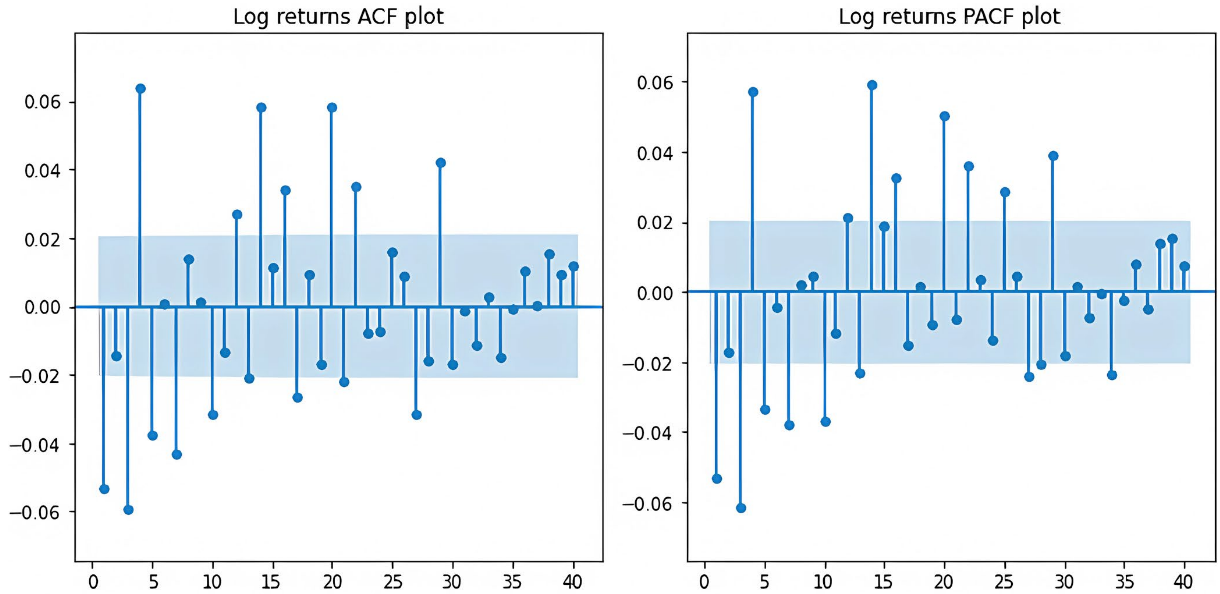

Figure 3. ACF and PACF plot.

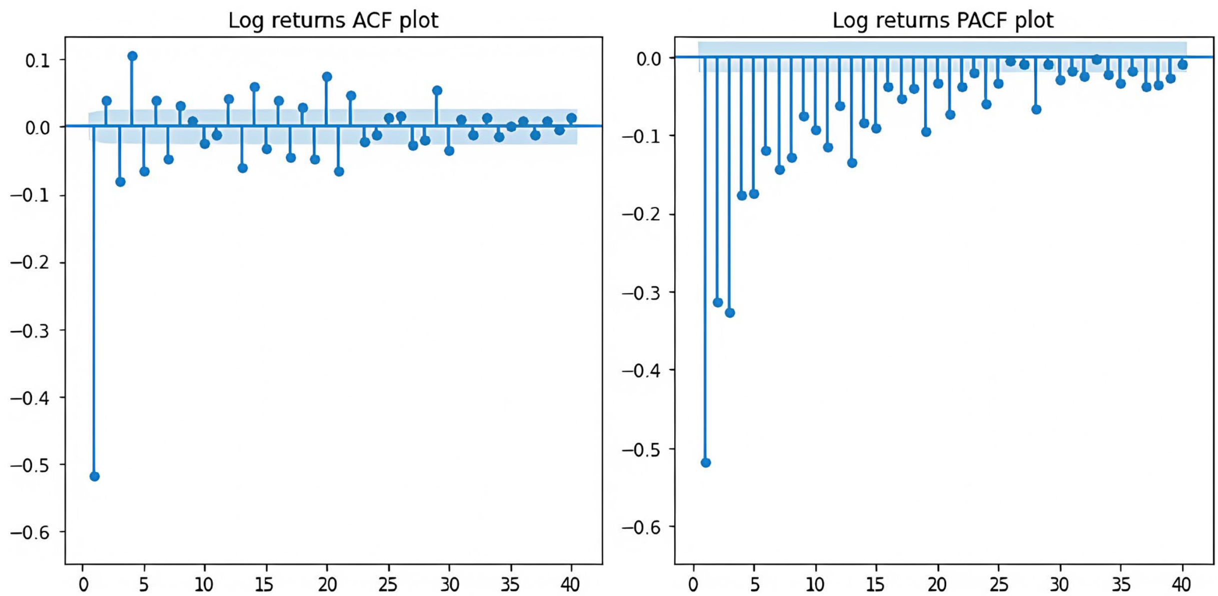

Figure 4. ACF and PACF plot of the differenced data.

3.3. Variance Autocorrelation Plots

Identification of the GARCH model is typically achieved through examination of the Autocorrelation Function (ACF) and Partial Autocorrelation Function (PACF) of the squared residuals. correlogram. From the squared correlogram where the highest ACF plot intersects with the lag is selected as p and where the highest PACF intersects with the lag line is selected as q. This gives a GARCH (p, q) model. Figure 5 and Figure 6 show the ACF and PACF plots of the variance equation, The series shows non-stationarity, implying additional differencing is required.

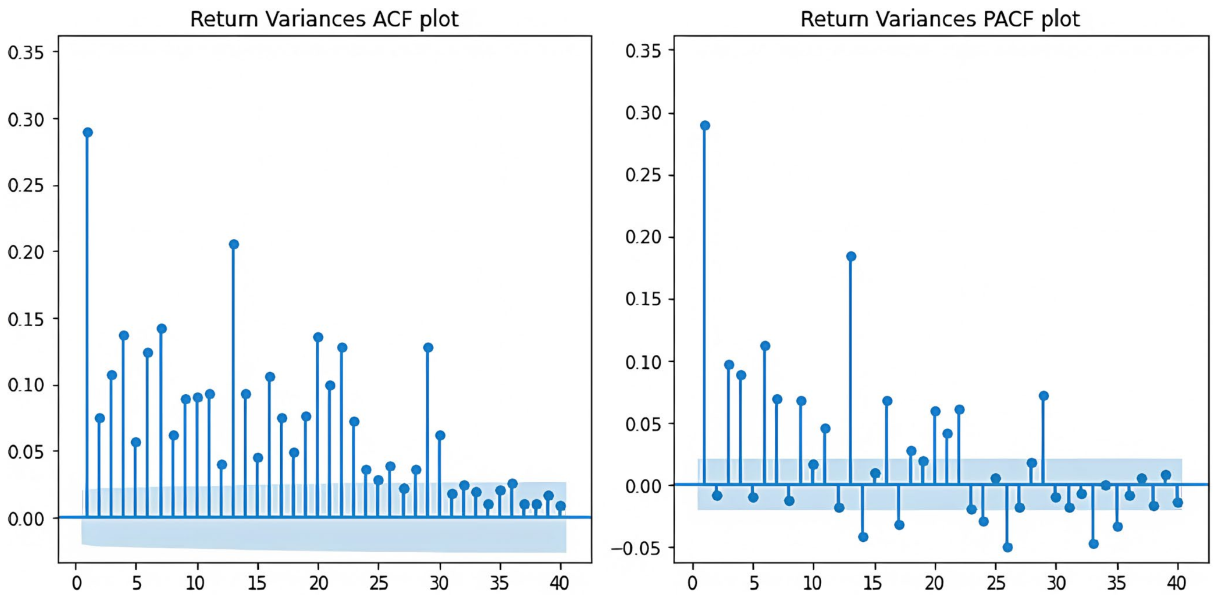

Figure 5. Variance of ACF plot.

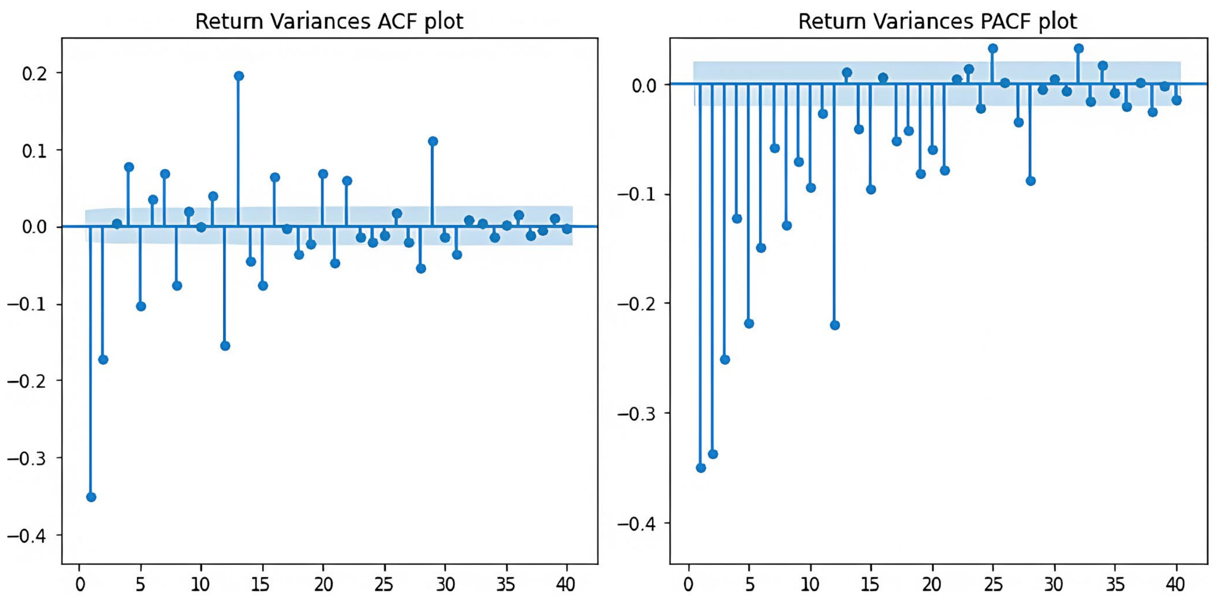

Figure 6. Return variances of differenced ACF and PACF plot.

The variance series is an ARIMA (2, 1, 4) process; thus, the log returns volatility is IGARCH (4, 4). The log returns process is ARIMA (0, 1, 1)-IGARCH (4, 4).

3.4. Data Fitting

Before fitting the financial data to the various distributions to find the Va estimates, of the various distributions used in the study that is; the normal, stable and generalized hyperbolic distributions were verified to determine whether the distributions could capture heavy tails since Value at Risk seeks to estimate the tails of an empirical distribution. Thus, **Figures 7-9** shows the density plots utilized to verify the tails of the distributions.

Commonly referred to as a "bell curve" due to its characteristic shape, the normal distribution exhibits symmetry about its mean, with a probability density function (PDF) that decays exponentially, yielding "thin" tails. Where the density plot shows that the normal distribution has the same tails as the log returns of oil prices as shown in **Figure 7**.

The tails of a stable distribution, unlike that of the normal distributions are "fat-tailed" in the sense that the density function decreases algebraically rather than exponentially for large data sets compared to the log returns of oil prices as shown in **Figure 8**.

The tails of the density function of the generalized hyperbolic distribution are "semi-heavy" because the probability density function decreases exponentially for large data sets but decreases slowly than the normal distribution. In fitting the data to the stable distributions, the data set is not wholly used since the distribution

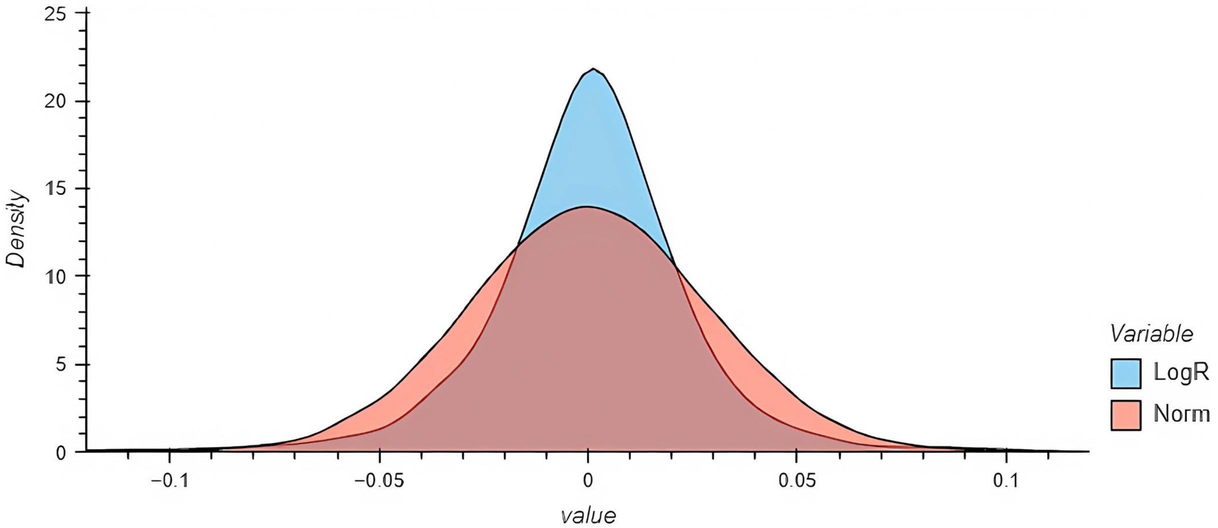

Figure 7. Density plot of normal distribution.

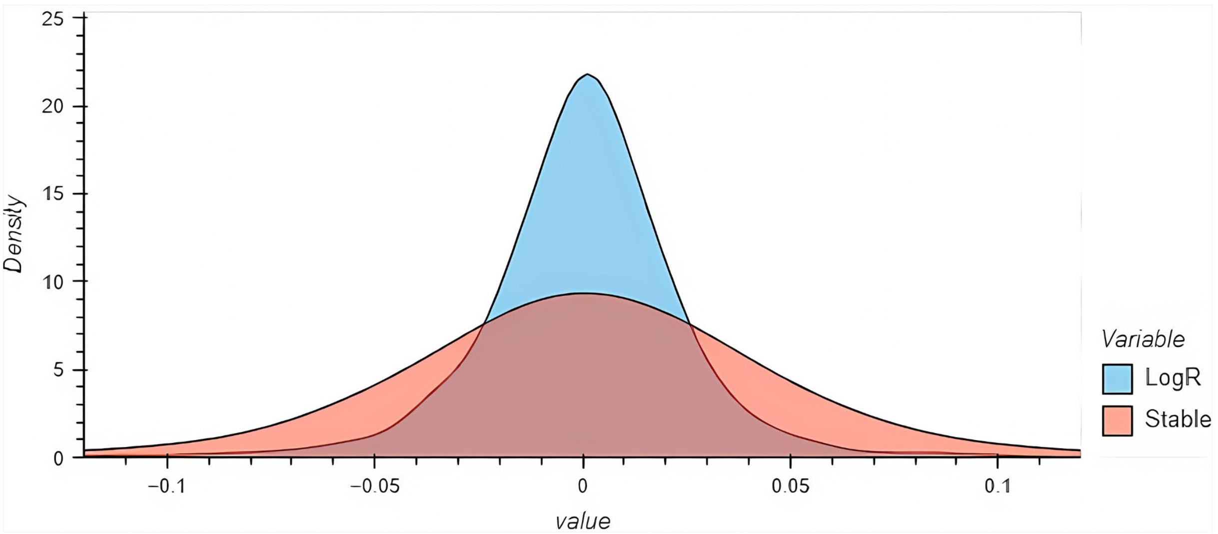

Figure 8. Density plot of stable distribution.

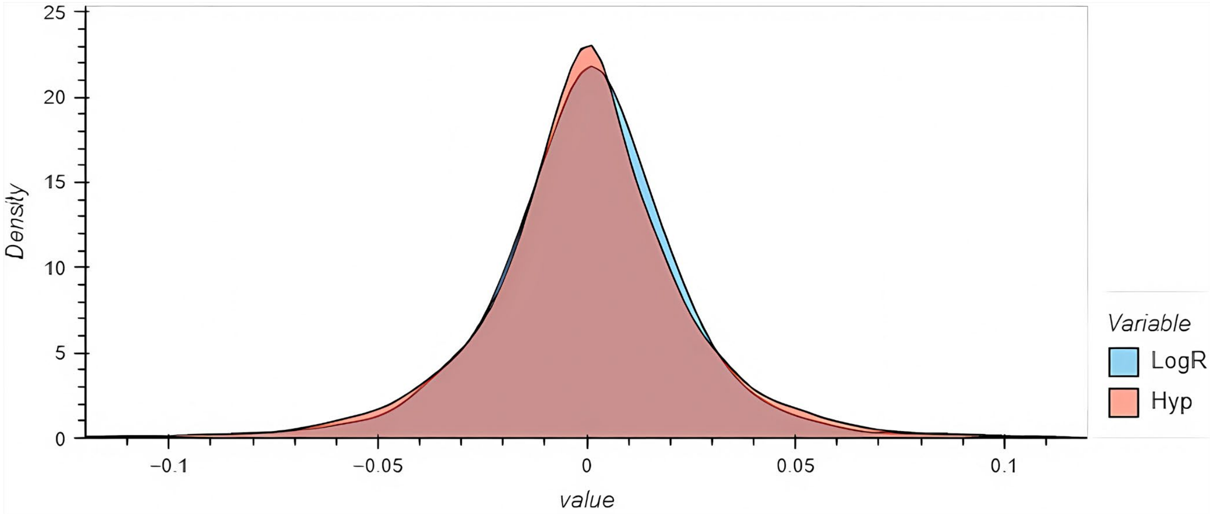

Figure 9. Density plot of generalized hyperbolic distribution.

does not have a general analytical CDF, nor does it have an analytical Probability Density Function (PDF). That way the algorithm fitting is very slow on CPU computer and thus, on the training set we are fitting on a few data points selected at random estimates the parameters of the characteristic equation.

Table 2 shows that stable distribution is poorly fitted to the data, that is not surprising given the visual plot comparing the empirical distribution of the data and a standard stable distribution. The coefficient of interest here is the one that reflects the kurtosis of the distribution it means alpha. Beta shows the skewness, and we see that our data distribution is fairly symmetric. The data fitted deduces a method to estimate the alpha values of the other data points of training set.

Table 2. Fitting data to stable distribution.

	Alpha	Beta	Loc	Scale	Stat	Critical value	P-value
0	1.630003	0.010545	0.001048	0.013243	0.725696	0.116489	0.011036
100	1.631195	0.013422	0.001055	0.01323	0.6297	0.116228	0.018908
150	1.640184	0.052775	0.001041	0.013334	0.505946	0.116946	0.038471
160	1.64369	0.047853	0.001002	0.013407	0.541502	0.116236	0.031297
175	1.637169	0.049651	0.001041	0.013422	0.572475	0.118405	0.02619
199	1.641512	0.057971	0.001162	0.013315	0.558793	0.116829	0.02833

3.5. One Day Ahead VaR Estimation at 95% and 99% Confidence Level

Table 3 summarizes the estimated generalized hyperbolic distribution parameters and goodness-of-fit test results for the VaR analysis. The VaR estimates for the data is shown in Table 4 and Table 5. In this study, three distributions were used for the parametric or variance-covariance method, that is, the Normal, Stable and Generalized Hyperbolic Distributions (GHD). In other studies, MonteCarlo and

Table 3. Fitting data to generalized hyperbolic distribution.

	p	a	b	Loc	Scale	Stat	Critical value	P_value
0	−1.90301	5.08E−01	−2.50E−01	0.004564	0.034373	2.07231	0.068761	0.000009
1	0.863388	2.86E−01	4.59E−04	0.000079	0.005199	1.011886	0.081398	0.002307
2	1.022333	2.05E−01	1.29E−03	−0.000091	0.003499	1.188415	0.08032	0.000896
3	−1.817265	3.46E−01	−1.91E−01	0.003794	0.032719	1.385977	0.071659	0.000314
4	1.128309	1.47E−01	1.64E−03	−0.00028	0.002392	1.390098	0.079597	0.000308
...	...	...	...	...	...	...	...	...
195	−1.353462	4.41E−07	2.94E−07	−0.000016	0.026026	1.24609	0.103029	0.000659
196	1.151027	1.39E−01	1.56E−03	−0.000275	0.002258	1.491958	0.079894	0.00018
197	−1.353256	5.77E−04	−2.13E−04	−0.000108	0.026031	1.573804	0.102926	0.000117
198	−0.621787	4.32E−01	−8.94E−04	0.000128	0.018925	1.015286	0.094667	0.002265
199	−1.395931	1.84E−05	1.69E−05	0.000113	0.02664	0.856001	0.100469	0.005378

Table 4. VAR estimates at 95% confidence level.

Day	Log R	Normal VaR	Stable VaR	GHD VaR	Monte Carlo VaR	Historical VaR
1	−0.001	−0.0323	−0.0364	−0.0432	−0.0459	0.0602
2	0.00938	−0.0313	−0.0364	−0.0401	−0.0447	−0.0396
3	0.01172	−0.0305	−0.0364	−0.0397	−0.0454	0.02061
4	0.00279	−0.0302	−0.0364	−0.0422	−0.0455	−0.0063
5	0.00688	−0.0296	−0.0364	−0.0422	−0.0452	−0.0283

Table 5. VAR estimates at 99% confidence level.

Day	Log R	Normal VaR	Stable VaR	GHD VaR	Monte Carlo VaR	Historical VaR
1	−0.001	−0.0458	−0.0789	−0.0768	−0.063	−0.0086
2	0.00938	−0.0444	−0.0789	−0.0685	−0.0643	0.00398
3	0.01172	−0.0433	−0.0789	−0.0672	−0.0623	0.01955
4	0.00279	−0.0428	−0.0789	−0.0773	−0.0641	0.00925
5	0.00688	−0.042	−0.0789	−0.066	−0.062	0.0273

Historical Simulations are utilized to estimate VaR, thus MonteCarlo and Historical Simulation methods were used to estimate VaR in addition to the parametric method for the purpose of comparing the VaR estimates.

Table 4 presents the One Day ahead VaR estimates of each distribution at 95% confidence level. Figure 10 is the graphical representation of the VaR estimates at 95% confidence level of the various distributions compared to the log returns of oil prices proposed from the graph that the further away the distribution is from the log returns, the better. When the log returns fall beneath the VaR estimate of the distribution a violation has occurred. From Figure 10 the violations of the normal distribution are smaller than any other distribution followed by the MonteCarlo simulations in that order.

Figure 11 shows violation that occurs in the VaR evaluation at 95% confidence level. The violation is the dispersion of the VaR estimate of the various distribution from the log returns of oil prices. The assumption made concerning the violation is that, the smaller the violation the better the distribution. This makes the normal distribution preferred above the other distributions used in the project.

Figure 12 depicts the VaR estimates at 95% confidence level of the various distributions compared to the log returns of oil prices violations of the stable, normal and GH distributions are smaller than any other distribution followed by the MonteCarlo simulations in that order. The violation for VaR at 99% confidence level are shown in Figure 13.

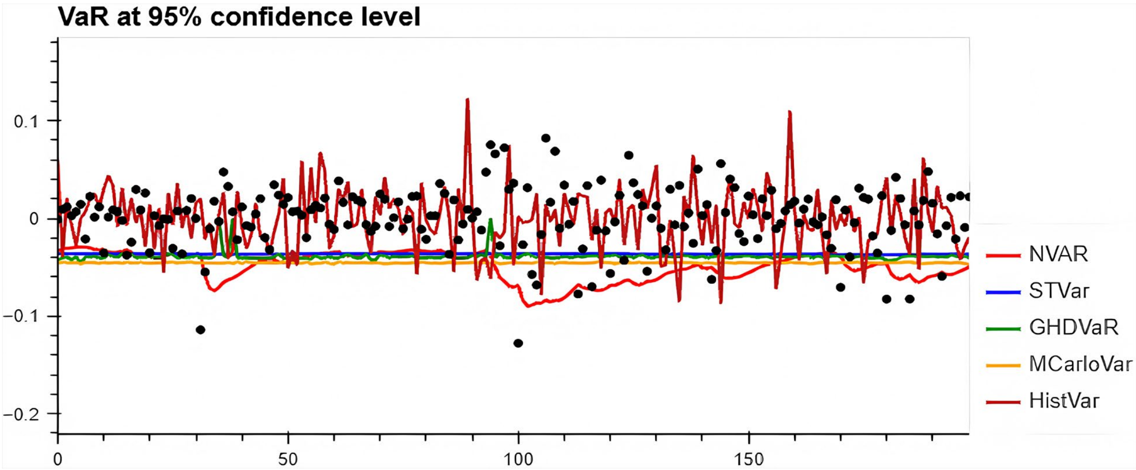

Figure 10. One day ahead VaR at 95% confidence level.

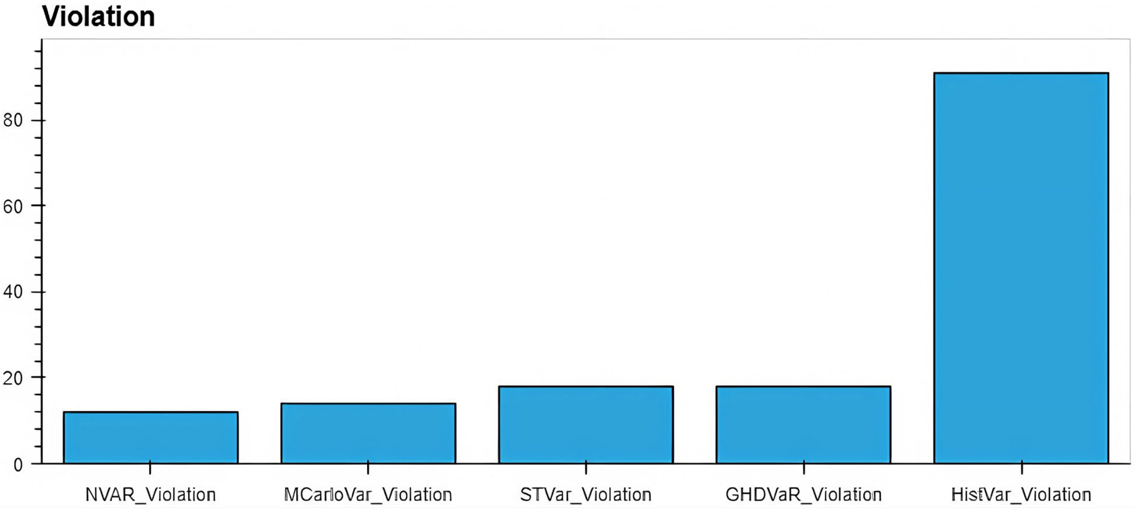

Figure 11. Violation for VaR at 95% confidence level.

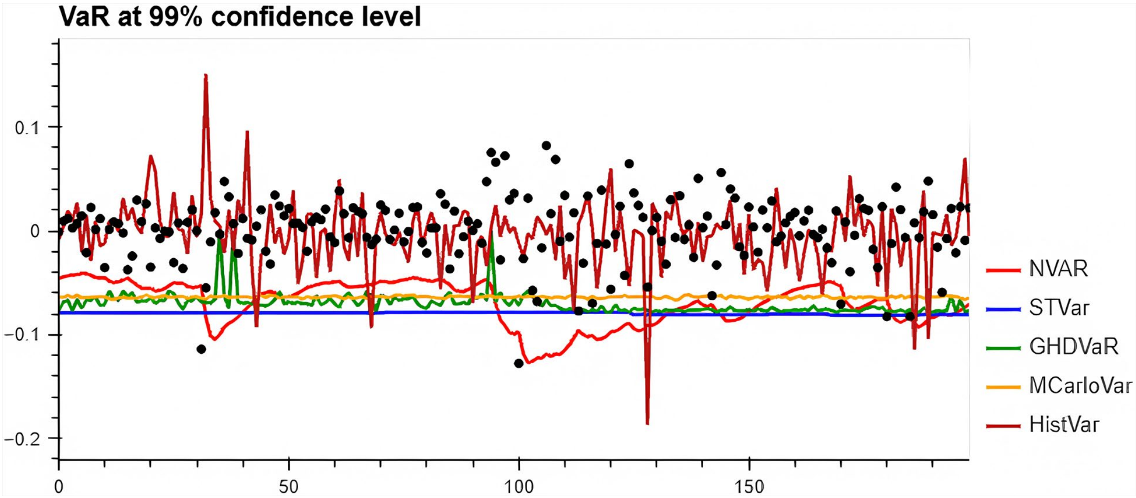

Figure 12. One day ahead VaR at 99% confidence level.

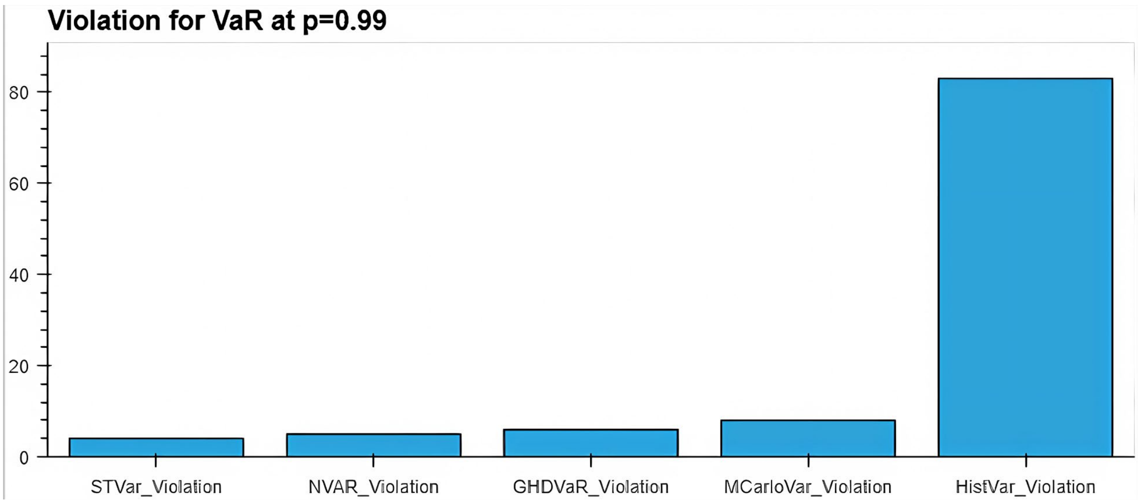

Figure 13. Violation for VaR at 99% confidence level.

3.6. Result of Kupiec Likelihood Ratio Test

The test statistics are asymptotically distributed as a chi-squared variate with one degree of freedom. Thus, the hypothesis test of the Kupiec test adopted in the study follows Table 6 and it is the hypothesis test for Chi-squared distribution.

Table 6. Chi-Square test (1).

Null hypothesis	H_0: Is accurate
Alternate hypothesis	H_1: Is not accurate

Decision: If the Critical value ≥ Test Statistic, Fail to reject H_0 else Reject H_0. From the statistical table, the Chi-Squared distribution at 95% confidence level, Critical value is 3.84.

Table 9 shows the distribution that gives accurate VaR estimates under the Kupiec backtesting at 95% confidence level. As evidence from Table 7, the test favours the normal distribution and MonteCarlo simulations, but the other distributions are rejected by the Kupiec test at 95% confidence level.

Also, for the Kupiec test at 99% confidence level; the Hypothesis test is shown in Table 7.

Table 7. Kupiec test at 95% confidence level.

	test stat	result at 95%
NVAR_Violation	0.418329	Fail to reject H_0
STVar_Violation	5.588523	Reject H_0
GHDVaR_Violation	5.588523	Reject H_0
MCarloVar_Violation	1.548963	Fail to reject H_0
HistVar_Violation	281.88408	Reject H_0

Table 8. Chi-square test (2).

Null hypothesis	H_0: Is accurate
Alternate hypothesis	H_1: Is not accurate

Decision: If the Critical value ≥ Test Statistic Fail to reject H_0 else Reject H_0 From the statistical table, the Chi-Squared distribution at 99% confidence level, Critical value is 6.63. Table 9 is the Kupiec test at 99% confidence level.

Table 9. Kupiec Statistic for VaR at 99% confidence level.

	test stat	result at 99%
NVAR_Violation	3.239257	Fail to reject H_0
STVar_Violation	1.585855	Fail to reject H_0
GHDVaR_Violation	5.305678	Fail to reject H_0
MCarloVar_Violation	10.426145	Reject H_0
HistVar_Violation	496.415074	Reject H_0

From Table 9, the normal, stable and generalized hyperbolic distributions were considered accurate whiles the MonteCarlo and Historical simulations were regarded as not accurate.

3.7. Result of Christoffersen Conditional Coverage Test

The test statistics are asymptotically distributed as a chi-squared variate with two degrees of freedom [32]. The hypothesis test of the Christoffersen test adopted in the study follows that; Hypothesis test shown in Table 10.

Table 10. Chi-square test (3).

Null hypothesis	H_0: Is accurate
Alternate hypothesis	H_1: Is not accurate

Decision: If the Critical value ≥ Test Statistic Fail to reject H0 else Reject H0 From the statistical table, the Chi-Squared distribution at 95% confidence level with two degrees of freedom has Critical value of 5.99.

Table 11. Christoffersen test statistic at 95% confidence level.

	Test stat	Result at 95%
NVAR_Violation	2.018922	Fail to reject H_0
STVar_Violation	23.316572	Reject H_0
GHDVaR_Violation	23.316572	Reject H_0
MCarloVar_Violation	31.151241	Reject H_0
HistVar_Violation	545.158654	Reject H_0

From **Table 11**, the VaR estimate for the normal distribution is considered to be accurate and the other distributions considered to be not accurate. From the statistical table, the Chi-Squared distribution at 99% confidence level with two degrees of freedom has Critical value of 11.34. **Table 12** reports the Christoffersen test statistics for the various distributions under consideration.

Table 12. Christoffersen conditional coverage test statistic for VaR at 99% confidence level.

	Test stat	Result at 99%
NVAR_Violation	3.501739	Fail to reject H_0
STVar_Violation	1.752528	Fail to reject H_0
GHDVaR_Violation	5.686663	Fail to reject H_0
MCarloVar_Violation	11.114423	Fail to reject H_0
HistVar_Violation	672.396206	Reject H_0

It is observed from the Table that, the normal, stable, generalized hyperbolic distributions and the MonteCarlo simulations have accurate VaR estimates whiles the Historical simulations were regarded as not accurate.

3.8. Results of Lopez Magnitude Loss Function

To compare different VaR models, the Lopez measure is used. It is a distance measure between the realized returns and the calculated VaR. A lower value of the Lopez distance indicates a superior distributional fit. Lopez magnitude loss function values are shown in **Table 13** and **Table 14**.

It can be observed from **Table 13** that the normal outperforms MonteCarlo and the MonteCarlo is better than Stable distribution. The stable distribution also performs better than the GHD in order of the magnitude of the Lopez distance.

The Lopez distance compares the different distributions and the smaller the distance, the better the distribution. Thus, it can be observed from **Table 13** that, Stable distribution performs better than the normal and the normal performs better than the GHD in order of magnitude of Lopez distance.

Table 13. Lopez distance at 95% confidence level.

	Lopez Loss
NVAR	11.987308
STVar	17.973731
GHDVaR	17.977025
MCarloVar	13.982033
HistVar	90.793746

Table 14. Lopez distance at 99% confidence level.

	Lopez Loss
NVAR	4.994749
STVar	3.996349
GHDVaR	5.994746
MCarloVar	7.992337
HistVar	82.782869

4. Conclusion

In this study, the variance-covariance technique was applied to estimate the VaR of OPEC oil prices. To capture the variability and uncertainty in the dataset and model the potential losses, three distributions were utilized: Normal, α-Stable, and Generalized Hyperbolic Distribution (GHD). All models were fitted to daily log-returns within an ARIMA-GARCH framework. To examine the efficacy of the estimated VaR, the Kupiec test, Christoffersen test, and Lopez's loss function were employed for backtesting. The results of the study indicate the following: At 95% Confidence level: The Kupiec test shows that ARIMA-GARCH-Normal and Monte Carlo are accurate, while ARIMA-GARCH-Stable, ARIMA-GARCH-GHD, and Historical Simulation are rejected. The Christoffersen test rejects all models except ARIMA-GARCH-Normal. Based on Lopez's loss function, the models rank from best to worst as: Normal, Monte Carlo, Stable, GHD, Historical Simulation. At 99% Confidence level: The Kupiec test and Christoffersen test indicate that ARIMA-GARCH-Normal, ARIMA-GARCH-Stable, and ARIMA-GARCH-GHD are statistically accurate. Based on Lopez's loss function, the models rank from best to worst as: Stable, Normal, GHD, Monte Carlo, Historical Simulation. While Normal, Stable, and GHD are all adequate at the 99% level, Lopez's loss function shows that ARIMA-GARCH-Stable and ARIMA-GARCH-Normal produce the lowest magnitude of errors. Therefore, ARIMA-GARCH-Normal is the most consistently accurate model across both confidence levels. It is recommended that risk analysts adopt the Normal and Stable distributions within an ARIMA-GARCH framework for the estimation of VaR for OPEC oil prices, as these distributions produced the most accurate and stable estimations across the different confidence levels and backtesting criteria.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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