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Synthetic Differential Geometry (SDG) of Stacks - Categorification of SDG

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Abstract

Orthodox synthetic differential geometry is concerned with microlinear spaces. This paper explains how to develop synthetic differential geometry of microlinear stacks or microlinear categories. As illustrations, we will address the categorical Lie algebra of infinitesimal endofunctors and infinitesimal natural transformations, the theory of differential forms and the categorified Ambrose-Palais-Singer theorem. In this paper, we use the word "stack" as a synonym of "category".

Keywords

Synthetic Differential Geometry, Categorification, Categorical Ambrose-Palais-Singer Theorem, Categorical Lie Algebra of Infinitesimal Endofunctors and Infinitesimal Natural Transformations, Categorical Differential Form

1. Introduction

Orthodox synthetic differential geometry (SDG) [1] [2] is concerned with spaces of a special kind, called *microlinear* spaces. This paper is concerned with differential geometry of microlinear stacks or microlinear categories. It is exactly established in orthodox SDG that any infinitesimal transformation is invertible, that is to say, bijective. This paper shows that any infinitesimal endofunctor is invertible, as is expected. However, we are also expected to show that any infinitesimal natural transformation between infinitesimal endofunctors is invertible. Invertible natural transformations with respect to vertical composition of natural transformations are usually called natural isomorphisms in the literature. We will show that any infinitesimal natural transformation between infinitesimal endofunctors is invertible with respect to horizontal composition of natural transformations. We will show in § 3 that the category of infinitesimal endofunctors and infinitesimal natural transformations of a microlinear category is a categorical Lie algebra, just as the space of infinitesimal transformations of a microlinear space forms a Lie algebra in orthodox SDG.

The Ambrose-Palais-Singer theorem [3], claiming the equivalence of symmetric connections and sprays, is in essence of categorical nature. In orthodox SDG, it was first established by Bunge and Sawyer [4] under certain hypotheses, having been established in general by Kock and Lavendhomme [5]. We will establish the categorical Ambrose-Palais-Singer theorem in § 5, where the proof is none other than a paraphrase of Kock and Lavendhomme's with strong emphasis on its categorical quintessence. § 4 addresses differential forms on the categorical level. We have confined our consideration to these three topics, but I expect that almost all the results of orthodox SDG can be elevated to the categorical level.

In universal algebra, an algebraic theory consists of a specification of operations and equational laws that these operations must satisfy. Models of the algebraic theory are called algebras. The notion of a monoid is a well-known example of an algebraic theory. A monoid is a set equipped with an associative binary operation and a unity element operating as both left and right unity. It is simple to categorify a given algebraic theory. By way of example, a *categorical monoid* is no other than an *internal* category within the category ***Mod*** of monoids, and their homomorphisms. Therefore, to make a category $\mathcal{M}$ a categorical monoid, we must make both $\mathrm{Ob}\,\mathcal{M}$ and $\mathrm{Mor}\,\mathcal{M}$ monoids so that such categorical mappings as $\mathrm{dom}:\mathrm{Mor}\,\mathcal{M}\to\mathrm{Ob}\,\mathcal{M}$, $\mathrm{cod}:\mathrm{Mor}\,\mathcal{M}\to\mathrm{Ob}\,\mathcal{M}$, $\mathrm{id}:\mathrm{Ob}\,\mathcal{M}\to\mathrm{Mor}\,\mathcal{M}$ and $\mathrm{comp}:\mathrm{Mor}\,\mathcal{M}\times_{\mathrm{Ob}\,\mathcal{M}}\mathrm{Mor}\,\mathcal{M}\to\mathrm{Mor}\,\mathcal{M}$ are homomorphisms of monoids, *i.e.*, mappings preserving binary operations and unity elements. Alternatively, a *categorical monoid* can be defined to be an *internal* monoid within the category ***Cat*** of small categories and functors. By way of example, a categorical monoid is a category $\mathcal{M}$ equipped with functors $F_2:\mathcal{M}\times\mathcal{M}\to\mathcal{M}$ and $F_0:1\to\mathcal{M}$ making the following diagrams commutative:

$$\begin{array}{ccc}
\mathcal{M}\times\mathcal{M}\times\mathcal{M} & \overset{F_2\times\mathrm{Id}_{\mathcal{M}}}{\rightarrow} & \mathcal{M}\times\mathcal{M} \\
{\scriptstyle \mathrm{Id}_{\mathcal{M}}\times F_2}\downarrow & & \downarrow{\scriptstyle F_2} \\
\mathcal{M}\times\mathcal{M} & \underset{F_2}{\rightarrow} & \mathcal{M}
\end{array}$$

$$\begin{array}{ccc}
\mathcal{M}\cong 1\times\mathcal{M} & \overset{F_0\times\mathrm{Id}_{\mathcal{M}}}{\rightarrow} & \mathcal{M}\times\mathcal{M} \\
{\scriptstyle \mathrm{Id}_{\mathcal{M}}}\searrow & & \swarrow{\scriptstyle F_2} \\
 & \mathcal{M} &
\end{array}$$

$$\begin{array}{ccc}
\mathcal{M}\cong \mathcal{M}\times 1 & \overset{\mathrm{Id}_{\mathcal{M}}\times F_0}{\rightarrow} & \mathcal{M}\times\mathcal{M} \\
{\scriptstyle \mathrm{Id}_{\mathcal{M}}}\searrow & & \swarrow{\scriptstyle F_2} \\
 & \mathcal{M} &
\end{array}$$

In general, any equational algebraic theory can be categorified by choosing functors corresponding to operations with commutative diagrams corresponding to equations.

The former approach is called the *concept-first-then categorification* approach, while the latter is called the *category-first-then-conceptualization* approach. It is well known that both approaches are essentially equivalent.

The setting in which we work is the same as that in [2]. In particular, $\boldsymbol{R}$ denotes the ring of line type which is supposed to abide by the general Kock-Lawvere axiom. We are slightly naive in referring to such a large category as $\boldsymbol{Cat}$, but we are content only to say that there are several methods to make such discussions legitimate. We do not hope to blur our basic geometrical idea by such secondary legitimizations.

2. Preliminary Considerations

Every set can be regarded as a discrete category. We denote the category of categories and functors and that of sets and mappings by $\boldsymbol{Cat}$ and $\mathbf{Set}$, respectively. Sets and *spaces* are regarded as synonyms. Categories and *stacks* are regarded as synonyms. Vertical and horizontal compositions of natural transformations α, β are denoted by $\beta \cdot \alpha$ and $\beta \circ \alpha$, respectively.

A natural transformation $\alpha : F \Rightarrow G$ is called a *vertical natural isomorphism* provided that there exists a natural transformation $\beta : G \Rightarrow F$ such that $\alpha \cdot \beta = \mathrm{id}$ and $\beta \cdot \alpha = \mathrm{id}$. This notion is simply called a natural isomorphism in the literature. A natural transformation $\alpha : F \Rightarrow G$ is called a *horizontal natural isomorphism* provided that there exists a natural transformation $\beta : F' \Rightarrow G'$ such that $\alpha \circ \beta = \mathrm{id}$ and $\beta \circ \alpha = \mathrm{id}$, which implies that all the compositions of functors $F' \circ F$, $F \circ F'$, $G' \circ G$ and $G \circ G'$ are the identity functors.

A functor $F : \mathcal{M} \to \mathcal{N}$ is called *isomorphic* if there exists a functor $G : \mathcal{N} \to \mathcal{M}$ such that $G \circ F = \mathrm{Id}_{\mathcal{M}}$ and $F \circ G = \mathrm{Id}_{\mathcal{N}}$.

It is well known that there is an adjunction

$$\mathrm{Hom}_{\boldsymbol{Cat}}\left(\mathcal{M} \times \mathcal{L}, \mathcal{N}\right) \cong \mathrm{Hom}_{\boldsymbol{Cat}}\left(\mathcal{M}, \mathcal{N}^{\mathcal{L}}\right)$$

for $\mathcal{L}, \mathcal{M}, \mathcal{N} \in \boldsymbol{Cat}$, By taking $\mathcal{L}$ to be a discrete category, *i.e.*, a set S, we have

$$\mathrm{Hom}_{\boldsymbol{Cat}}\left(\mathcal{M} \times S, \mathcal{N}\right) \cong \mathrm{Hom}_{\boldsymbol{Cat}}\left(\mathcal{M}, \mathcal{N}^{S}\right)$$

with $\mathcal{M} \times S = \coprod_S \mathcal{M}$ and $\mathcal{N}^S = \prod_S \mathcal{N}$. This means that $\boldsymbol{Cat}$ is tensored and cotensored over $\mathbf{Set}$.

An *internal category* $\mathcal{M}$ within a complete category $\mathcal{C}$ is a pair $\left(\mathrm{Ob}\,\mathcal{M}, \mathrm{Mor}\,\mathcal{M}\right)$ of objects in $\mathcal{C}$ along with morphisms in $\mathcal{C}$

$$\mathrm{dom} : \mathrm{Mor}\,\mathcal{M} \to \mathrm{Ob}\,\mathcal{M}$$

$$\mathrm{cod} : \mathrm{Mor}\,\mathcal{M} \to \mathrm{Ob}\,\mathcal{M}$$

$$\mathrm{id} : \mathrm{Ob}\,\mathcal{M} \to \mathrm{Mor}\,\mathcal{M}$$

$$\mathrm{comp} : \mathrm{Mor}\,\mathcal{M} \times_{\mathrm{Ob}\,\mathcal{M}} \mathrm{Mor}\,\mathcal{M} \to \mathrm{Mor}\,\mathcal{M}$$

abiding by the usual axioms such as associativity of composition.

A ring of line type $\boldsymbol{R}$ is fixed. We recall some infinitesimal spaces.

$$D_0 = \{0\}$$
$$D = D_1 = \{d \in \mathbf{R} \mid d^2 = 0\}$$
$$D_k = \{e \in \mathbf{R} \mid e^{k+1} = 0\}$$
$$D(n) = \{(d_1, \cdots, d_n) \in D^n \mid d_i d_j = 0 \text{ for any } i \neq j\}$$
$$D_k(n) = \{(e_1, \cdots, e_n) \in (D_k)^n \mid e_i e_j = 0 \text{ for any } i \neq j\}$$
$$D^2(3) = \{(d_1, d_2, d_1', d_2', d_1'', d_2'') \in (D^2)^3 \mid d_i d_j' = d_i' d_j'' = d_i d_j'' = 0 \text{ for any } i, j = 1, 2\}$$

The category of infinitesimal spaces, which is the dual of the category of Weil algebras, is of coproduct $\oplus$ and an initial object D_0 By way of example, we have

$$D^2(3) = D^2 \oplus D^2 \oplus D^2$$
$$D(4) = D \oplus D \oplus D \oplus D$$

The following quasi-colimit diagrams will be used implicitly in frequence.

$$\begin{array}{ccc} D^2 & \xrightarrow{(d_1,d_2)\in D^2 \mapsto d_1+d_2 \in D} & D \\ {\scriptstyle d_1,d_2 \in D^2 \mapsto d_1+d_2\in D}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_D} \\ D & \xrightarrow[\mathrm{Id}_D]{} & D \end{array}$$

$$\begin{array}{ccc} D_m \times D & \xrightarrow{(e,d)\in D_m \times D \mapsto ed \in D} & D \\ {\scriptstyle e,d \in D_m \times D \mapsto ed\in D}\downarrow & & \downarrow{\scriptstyle \mathrm{Id}_D} \\ D & \xrightarrow[\mathrm{Id}_D]{} & D \end{array}$$

A category $\mathcal{M}$ is called *microlinear* if $\mathcal{M}^{\boldsymbol{D}}$ is a limit diagram in ***Cat*** for any finite quasi colimit diagram $\boldsymbol{D}$ of infinitesimal spaces. It is easy to see that

Proposition 2.1. *Any limit of microlinear categories in* **Cat** *is microlinear. If* $\mathcal{M}$ *is a microlinear category and* $\mathcal{N}$ *is a category, then* $\mathcal{M}^{\mathcal{N}}$ *is a microlinear category. In particular, if* $\mathcal{M}$ *is a microlinear category and* S *is a set, then* $\mathcal{M}^S$ *is a microlinear category.*

Proof. The first statement follows from the interchangeability of two limits. The second statemnt follows from the exponential law $(\mathcal{M}^{\mathcal{N}})^S = (\mathcal{M}^S)^{\mathcal{N}}$ for any set S and the fact that the functor $(\_)^N : \boldsymbol{Cat} \to \boldsymbol{Cat}$ preserves limits. □

Since the forgetful functor $\boldsymbol{Cat} \to \boldsymbol{Set} \times \boldsymbol{Set}$ preserves and creates limits, we have

Theorem 2.2. *A microlinear category is no other than an internal category within the category of microlinear spaces and mappings.*

In this paper, $\mathcal{M}$ will always be a microlinear category.

3. The Categorical Lie Algebra of Infinitesimal Endofunctors and Infinitesimal Natural Transformations

We will show that the functor $\pi_0 : \mathcal{M}^D \to \mathcal{M}$ assigning the object at 0 to each object in $\mathcal{M}^D$ and the morphism at 0 to each morphism in $\mathcal{M}^D$ a *categorical* $\boldsymbol{R}$*-module fibered over* $\mathcal{M}$ in the sense that it is an internal $\boldsymbol{R}$-module within the slice category $\boldsymbol{Cat}/\mathcal{M}$.

Now we will define functors for a categorical $\boldsymbol{R}$-module fibered over $\mathcal{M}$ corresponding to scalar multiplication, addition, zero with respect to addition and inverse with respect to addition.

- For each $r \in \boldsymbol{R}$, the functor
$$\mathcal{M}^{d\in D\mapsto rd\in D} : \mathcal{M}^D \to \mathcal{M}^D$$
is denoted by $\boldsymbol{D}^r$.
- The functor
$$\mathcal{M}^{d\in D\mapsto (d,d)\in D(2)} : \begin{array}{c} \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \\ \cong \\ \mathcal{M}^{D(2)} \end{array} \to \mathcal{M}^D$$

is denoted by $\boldsymbol{D}^+$, where the isomorphism $\mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \cong \mathcal{M}^{D(2)}$ comes from the quasi-colimit diagram
$$\begin{array}{ccc} D_0 & \xrightarrow{i} & D \\ {}_i\downarrow & & \downarrow_{d\in D\mapsto (d,0)\in D(2)} \\ D & \underset{d\in D\mapsto (0,d)\in D(2)}{\rightarrow} & D(2) \end{array}$$

- The functor
$$\mathcal{M}^{d\in D\mapsto 0} : \begin{array}{c} \mathcal{M} \\ \cong \\ \mathcal{M}^{D_0} \end{array} \to \mathcal{M}^D$$

is denoted by $\boldsymbol{D}^0$.
- The functor
$$\mathcal{M}^{d\in D\mapsto -d\in D} : \mathcal{M}^D \to \mathcal{M}^D$$

is denoted by $\boldsymbol{D}^-$.

Now we will show that the above four functors make $\mathcal{M}^D$ a categorical $\boldsymbol{R}$-module fibered over $\mathcal{M}$.

Theorem 3.1. *The functor* $\pi_0 : \mathcal{M}^D \to \mathcal{M}$ *is a categorical* $\boldsymbol{R}$*-module fibered over* $\mathcal{M}$.

Proof. We have to make sure that the four functors corresponding to scalar multiplication, addition, zero with respect to addition and inverse with respect to addition abide by the usual equational axioms of $\boldsymbol{R}$-module. Here we only deal with the associativity of addition, which follows from the commutativity of the following diagram:
$$\begin{array}{ccccc} \mathcal{M}^{D(2)} \times_{\mathcal{M}} \mathcal{M}^D & \cong & \mathcal{M}^{D(3)} & \cong & \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^{D(2)} \\ {}_{\boldsymbol{D}^+ \times_{\mathcal{M}} \mathrm{Id}_{\mathcal{M}^D}}\downarrow & & & & \downarrow_{\mathrm{Id}_{\mathcal{M}^D} \times_{\mathcal{M}} \boldsymbol{D}^+} \\ \mathcal{M}^{D(2)} & \underset{\boldsymbol{D}+}{\rightarrow} & \mathcal{M}^D & \underset{\boldsymbol{D}+}{\leftarrow} & \mathcal{M}^{D(2)} \end{array}$$

□

Corollary 3.2. *For any subcategory* $\mathcal{M}' \subset \mathcal{M}$, *the restriction of the functor* π_0 *to* $(\pi_0)^{-1}(\mathcal{M}') \to \mathcal{M}'$ *is a categorical* $\boldsymbol{R}$*-module fibered over* $\mathcal{M}'$. *In particular, for any object* $a \in \mathcal{M}$, *the category* $(\mathcal{M}^D)_a = (\pi_0)^{-1}(\{a, \mathrm{id}_a\})$ *is a categorical* $\boldsymbol{R}$*-module.*

Now we will show that the categorical $\boldsymbol{R}$-module $\pi_0 : \mathcal{M}^D \to \mathcal{M}$ fibered over $\mathcal{M}$ is *Euclidean* in the sense that there is a unique functor $\mathcal{M}^{D^2} \to \mathcal{M}^D$ making the diagram

$$\begin{array}{ccc} \left(\mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^{D_0}\right)^D \cong \mathcal{M}^{D^2} & \to & \mathcal{M}^D \\ \downarrow{\scriptstyle \left(\mathrm{Id}_{\mathcal{M}^D} \times \boldsymbol{D}^- \circ \boldsymbol{D}^0\right)^D} & & \downarrow{\scriptstyle \mathcal{M}^{(d_1,d_2)\in D^2 \mapsto d_1 d_2 \in D}} \\ \left(\mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D\right)^D & \underset{\left(\boldsymbol{D}^+\right)^D}{\to} & \left(\mathcal{M}^D\right)^D \cong \mathcal{M}^{D^2} \end{array}$$

commutative. Generally, a categorical $\boldsymbol{R}$-module $\mathcal{V}$ is called *Euclidean* if there exists a unique functor $\varphi : \mathcal{V}^D \to \mathcal{V}$ with

$$\pi_d = \pi_0 + d\varphi$$

where π_d is the standard projection $\mathcal{V}^D \to \mathcal{V}$ assigning the value at d to each entity in $\mathcal{V}^D$. It is easy to see that a categorical $\boldsymbol{R}$-module $\mathcal{V}$ is Euclidean if both the induced $\boldsymbol{R}$-module $\mathrm{Ob}\,\mathcal{V}$ and the induced $\boldsymbol{R}$-module $\mathrm{Mor}\,\mathcal{V}$ are Euclidean in the classical sense. It is also easy to see that if $\mathcal{V}$ is a Euclidean categorical $\boldsymbol{R}$-module and $\mathcal{N}$ is a category, then $\mathcal{V}^{\mathcal{N}}$ is a Euclidean categorical $\boldsymbol{R}$-module.

Theorem 3.3. *The categorical* $\boldsymbol{R}$*-module* $\pi_0 : \mathcal{M}^D \to \mathcal{M}$ *fibered over* $\mathcal{M}$ *is Euclidean.*

Proof. This follows from the quasi-colimit diagram

$$D \begin{array}{c} \overset{d\in D \mapsto (d,0)\in D^2}{\to} \\ \overset{d\in D \mapsto (0,d)\in D^2}{\to} \\ \overset{d\in D \mapsto (0,0)\in D^2}{\to} \end{array} D^2 \overset{(d_1,d_2)\in D^2 \mapsto d_1 d_2 \in D}{\to} D$$

and the commutativity of the diagram

$$\left(\mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^{D_0}\right)^D \overset{\left(\mathrm{Id}_{\mathcal{M}^D} \times \boldsymbol{D}^- \circ \boldsymbol{D}^0\right)^D}{\to} \left(\mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D\right)^D \overset{\left(\boldsymbol{D}^+\right)^D}{\to} \left(\mathcal{M}^D\right)^D \begin{array}{c} \overset{\mathcal{M}^{d\in D \mapsto (d,0)\in D^2}}{\to} \\ \overset{\mathcal{M}^{d\in D \mapsto (0,d)\in D^2}}{\to} \\ \overset{\mathcal{M}^{d\in D \mapsto (0,0)\in D^2}}{\to} \end{array} \mathcal{M}^D$$

Now we would like to make the categorical $\boldsymbol{R}$-module $\left(\left(\mathcal{M}^{\mathcal{M}}\right)^D\right)_{\mathrm{Id}_{\mathcal{M}}}$ a categorical Lie $\boldsymbol{R}$-algebra. It is easy to see that

Proposition 3.4. *We have*

$$\boldsymbol{D}^+(F,G)_d = G_d \circ F_d = F_d \circ G_d$$

for any $(F,G) \in \mathrm{Ob}\left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}} \times \left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}}$ *and any* $d \in D$ *as well as*

$$\boldsymbol{D}^{+}(\alpha,\beta)_{d} = \beta_{d} \circ \alpha_{d} = \alpha_{d} \circ \beta_{d}$$

for any $(\alpha,\beta) \in \mathrm{Mor}\left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}} \times \left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}}$ *and any* $d \in D$, *where* $\circ$ *stands for the composition of functors in the former formula, while* $\circ$ *stands for the horizontal composition of natural transformations in the latter formula.*

Corollary 3.5. *For any* $F \in \mathrm{Ob}\left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}}$ *and any* $d \in D$, F_d *is an isomorphic endofunctor, while, for any* $\alpha \in \mathrm{Mor}\left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}}$ *and any* $d \in D$, α_d *is a horizontal natural isomorphism.*

We define a functor $\boldsymbol{D}^{\mathrm{lie}} : \left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}} \times \left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}} \to \left(\mathcal{M}^{\mathcal{M}}\right)^{D^2}$ by the formulas

$$(F,G) \in \mathrm{Ob}\left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}} \times \left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}}$$

$$\mapsto \left((d_1,d_2) \in D^2 \mapsto G_{-d_2} \circ F_{-d_1} \circ G_{d_2} \circ F_{d_1}\right) \in \mathrm{Ob}\left(\mathcal{M}^{\mathcal{M}}\right)^{D^2}$$

$$(\alpha,\beta) \in \mathrm{Mor}\left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}} \times \left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}}$$

$$\mapsto \left((d_1,d_2) \in D^2 \mapsto \beta_{-d_2} \circ \alpha_{-d_1} \circ \beta_{d_2} \circ \alpha_{d_1}\right) \in \mathrm{Mor}\left(\mathcal{M}^{\mathcal{M}}\right)^{D^2}$$

where $\circ$ stands for the composition of functors in the former formula, while $\circ$ stands for the horizontal composition of natural transformations in the latter formula.

Proposition 3.6. *There is a unique functor*
$\boldsymbol{D}^{\mathrm{Lie}} : \left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}} \times \left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}} \to \left(\mathcal{M}^{\mathcal{M}}\right)^{D}$ *making the diagram*

$$\begin{tikzcd}
\left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}} \times \left(\left(\mathcal{M}^{\mathcal{M}}\right)^{D}\right)_{\mathrm{Id}_{\mathcal{M}}} \arrow[d, "\boldsymbol{D}^{\mathrm{lie}}"'] \arrow[dr, "\boldsymbol{D}^{\mathrm{lie}}"] & \\
\left(\mathcal{M}^{\mathcal{M}}\right)^{D^2} & \left(\mathcal{M}^{\mathcal{M}}\right)^{D} \arrow[l, "\left(\mathcal{M}^{\mathcal{M}}\right)^{(d_1,d_2)\in D^2 \mapsto d_1 d_2 \in D}"']
\end{tikzcd}$$

commutative.

Proof. This follows from the quasi-colimit diagram

$$D \underset{d\in D\mapsto(0,0)\in D^2}{\overset{\substack{d\in D\mapsto(d,0)\in D^2 \\ \to \\ d\in D\mapsto(0,d)\in D^2}}{\to}} D^2 \xrightarrow{(d_1,d_2)\in D^2 \mapsto d_1 d_2 \in D} D$$

and the fact that the diagram

$$\left(\left(\mathcal{M}^{\mathcal{M}}\right)^D\right)_{\mathrm{Id}_{\mathcal{M}}} \times \left(\left(\mathcal{M}^{\mathcal{M}}\right)^D\right)_{\mathrm{Id}_{\mathcal{M}}} \xrightarrow{\boldsymbol{D}^{\mathrm{lie}}} \left(\mathcal{M}^{\mathcal{M}}\right)^{D^2} \begin{array}{c} \left(\mathcal{M}^{\mathcal{M}}\right)^{d\in D\mapsto(d,0)\in D^2} \\ \left(\mathcal{M}^{\mathcal{M}}\right)^{d\in D\mapsto(0,d)\in D^2} \\ \left(\mathcal{M}^{\mathcal{M}}\right)^{d\in D\mapsto(0,0)\in D^2} \end{array} \left(\mathcal{M}^{\mathcal{M}}\right)^D$$

is commutative. □

Now we have

Theorem 3.7. $\left(\left(\mathcal{M}^{\mathcal{M}}\right)^D\right)_{\mathrm{Id}_{\mathcal{M}}}$ *is a categorical Lie* $\boldsymbol{R}$*-algebra.*

The proof is deferred to the **Appendix**.

4. Categorical Differential Forms

In this section $\mathcal{E}$ will always be a Euclidean categorical $\boldsymbol{R}$-module which is microlinear as a category.

We define three functors:

(1) For any integer i between 1 and n and any $r\in\boldsymbol{R}$, we define a functor $\boldsymbol{D}_r^i:\mathcal{M}^{D^n}\to\mathcal{M}^{D^n}$ to be

$$\left(\left(d_1,\cdots,d_n\right)\in D^n\mapsto x_{d_1,\cdots,d_n}\in\mathrm{Ob}\,\mathcal{M}\right)\in\mathrm{Ob}\left(\mathcal{M}^{D^n}\right)\mapsto$$

$$\left(\left(d_1,\cdots,d_n\right)\in D^n\mapsto x_{d_1,\cdots,rd_i,\cdots,d_n}\in\mathrm{Ob}\,\mathcal{M}\right)\in\mathrm{Ob}\left(\mathcal{M}^{D^n}\right)$$

$$\left(\left(d_1,\cdots,d_n\right)\in D^n\mapsto f_{d_1,\cdots,d_n}\in\mathrm{Mor}\,\mathcal{M}\right)\in\mathrm{Mor}\left(\mathcal{M}^{D^n}\right)\mapsto$$

$$\left(\left(d_1,\cdots,d_n\right)\in D^n\mapsto f_{d_1,\cdots,rd_i,\cdots,d_n}\in\mathrm{Mor}\,\mathcal{M}\right)\in\mathrm{Mor}\left(\mathcal{M}^{D^n}\right)$$

(2) For any permutation σ of the numbers $1,2,\cdots,n$, we define a functor $\boldsymbol{D}^\sigma:\mathcal{M}^{D^n}\to\mathcal{M}^{D^n}$ to be

$$\left(\left(d_1,\cdots,d_n\right)\in D^n\mapsto x_{d_1,\cdots,d_n}\in\mathrm{Ob}\,\mathcal{M}\right)\in\mathrm{Ob}\left(\mathcal{M}^{D^n}\right)\mapsto$$

$$\left(\left(d_1,\cdots,d_n\right)\in D^n\mapsto x_{d_{\sigma(1)},\cdots,d_{\sigma(n)}}\in\mathrm{Ob}\,\mathcal{M}\right)\in\mathrm{Ob}\left(\mathcal{M}^{D^n}\right)$$

$$\left(\left(d_1,\cdots,d_n\right)\in D^n\mapsto f_{d_1,\cdots,d_n}\in\mathrm{Mor}\,\mathcal{M}\right)\in\mathrm{Mor}\left(\mathcal{M}^{D^n}\right)\mapsto$$

$$\left(\left(d_1,\cdots,d_n\right)\in D^n\mapsto f_{d_{\sigma(1)},\cdots,d_{\sigma(n)}}\in\mathrm{Mor}\,\mathcal{M}\right)\in\mathrm{Mor}\left(\mathcal{M}^{D^n}\right)$$

(3) For any $e\in D$ and any integer i between 1 and $n+1$, we define a functor $\boldsymbol{F}_e^i:\mathcal{M}^{D^{n+1}}\to\mathcal{M}^{D^n}$ by

$$\left(\left(d_1,\cdots,d_{n+1}\right)\in D^{n+1}\mapsto x_{d_1,\cdots,d_{n+1}}\in\mathrm{Ob}\,\mathcal{M}\right)\in\mathrm{Ob}\left(\mathcal{M}^{D^{n+1}}\right)\mapsto$$

$$\left(\left(d_1,\cdots,\hat{d}_i,\cdots,d_{n+1}\right)\in D^n\mapsto x_{d_1,\cdots,\underset{i}{e},\cdots,d_{n+1}}\in\mathrm{Ob}\,\mathcal{M}\right)\in\mathrm{Ob}\left(\mathcal{M}^{D^n}\right)$$

$$\left(\left(d_1,\cdots,d_{n+1}\right)\in D^{n+1}\mapsto f_{d_1,\cdots,d_{n+1}}\in\mathrm{Mor}\,\mathcal{M}\right)\in\mathrm{Mor}\left(\mathcal{M}^{D^{n+1}}\right)\mapsto$$

$$\left(\left(d_1,\cdots,\hat{d}_i,\cdots,d_{n+1}\right)\in D^n\mapsto f_{d_1,\cdots,\underset{i}{e},\cdots,d_{n+1}}\in\mathrm{Mor}\,\mathcal{M}\right)\in\mathrm{Mor}\left(\mathcal{M}^{D^n}\right)$$

A *categorical differential* n *-form on* $\mathcal{M}$ *with values in* $\mathcal{E}$ is a functor $\omega : \mathcal{M}^{D^n} \to \mathcal{E}$ abiding by the following two conditions:

(1) For any integer i between 1 and n and any $r \in \boldsymbol{R}$, we have

$$\omega \circ \mathbf{D}_r^i = r\omega$$

(2) For any permutation σ of the numbers $1,2,\cdots,n$, we have

$$\omega \circ \mathbf{D}^\sigma = \varepsilon_\sigma \omega$$

where ε_σ denotes the sign of the permutation σ.

Theorem 4.1. *Let* $\varphi : \mathcal{M}^{D^n} \times D^n \to \mathcal{E}$ *be a functor abiding by the following conditions*:

(1) *For each* $(e_1,\cdots,e_n) \in D^n$, *the functor* $\varphi(-,e_1,\cdots,e_n) : \mathcal{M}^{D^n} \to \mathcal{E}$ *is a categorical differential* n *-form on* M *with values in* E;

(2) *For each* $f \in \operatorname{Mor}\mathcal{M}^{D^n}$, *the mapping* $\varphi(f,-) : D^n \to \operatorname{Mor}\mathcal{E}$ *is homogeneous component wise in the sense that*

$$\varphi(f,e,\cdots,re_i,\cdots,e_n) = r\varphi(f,e_1,\cdots,e_i,\cdots,e_n)$$

for each $r \in \boldsymbol{R}$ *and each integer* i *between 1 and* n.

Then there is a unique categorical differential n *-form* $\omega : \mathcal{M}^{D^n} \to \mathcal{E}$ *with values in* $\mathcal{E}$ *such that*

$$\varphi(-,e_1,\cdots,e_n) = e_1 \cdots e_n \omega(-)$$

Proof. The proof is based on the quasi-colimit diagram

$$D^{n-1} \begin{array}{c} \xrightarrow{(d,\cdots,d_{n-1}) \mapsto (0,d,\cdots,d_{n-1})} \\ \vdots \\ \xrightarrow{(d,\cdots,d_{n-1}) \mapsto (d_1,\cdots,d_{i-1},0,d,\cdots,d_{n-1})} \\ \vdots \\ \xrightarrow{(d,\cdots,d_{n-1}) \mapsto (d,\cdots,d_{n-1},0)} \\ \xrightarrow{(d,\cdots,d_{n-1}) \mapsto (0,\cdots,0)} \end{array} D^n \xrightarrow{(d,\cdots,d_n) \in D^n \mapsto d \cdots d_n \in D} D$$

Since $\operatorname{Ob}\mathcal{E}^{\mathcal{M}^{D^n}}$ is a microlinear space and the diagram

$$D^{n-1} \begin{array}{c} \xrightarrow{(d,\cdots,d_{n-1}) \mapsto (0,d,\cdots,d_{n-1})} \\ \vdots \\ \xrightarrow{(d,\cdots,d_{n-1}) \mapsto (d_1,\cdots,d_{i-1},0,d,\cdots,d_{n-1})} \\ \vdots \\ \xrightarrow{(d,\cdots,d_{n-1}) \mapsto (d,\cdots,d_{n-1},0)} \\ \xrightarrow{(d,\cdots,d_{n-1}) \mapsto (0,\cdots,0)} \end{array} D^n \xrightarrow{\overline{\varphi}} \operatorname{Ob}\mathcal{E}^{\mathcal{M}^{D^n}}$$

equalizes for the mapping $\overline{\varphi} : D^n \to \operatorname{Ob}\mathcal{E}^{\mathcal{M}^{D^n}}$ associated to φ, there exists a unique $\overline{\omega} : D \to \operatorname{Ob}\mathcal{E}^{\mathcal{M}^{D^n}}$ with

$$\overline{\omega} \circ \left((d,\cdots,d_n) \in D^n \mapsto d \cdots d_n \in D \right) = \overline{\varphi}$$

Since $\mathrm{Ob}\,\mathcal{E}^{\mathcal{M}^{D^n}}$ is an Euclidean $\boldsymbol{R}$-module, there is a unique $\omega \in \mathrm{Ob}\,\mathcal{E}^{\mathcal{M}^{D^n}}$ with $\varpi(d) = d\omega$ for any $d \in D$. It is easy to see that is the desired categorical differential n-form on $\mathcal{M}$. □

Corollary 4.2. *Given a categorical differential* n *-form* ω *with values in a categorical Euclidean* $\boldsymbol{R}$ *-module* $\mathcal{E}$, *the functor* $\tilde{\omega}'\mathcal{M}^{D^{n+1}} \times D^{n+1} \to \mathcal{E}$ *defined by the formula*

$$\sum_{i=1}^{n+1} (-1)^i e_1 \cdots \hat{e}_i \cdots e_{n+1} \left(\omega \circ \boldsymbol{F}_0^i\right) - \sum_{i=1}^{n+1} (-1)^i e_1 \cdots \hat{e}_i \cdots e_{n+1} \left(\omega \circ \boldsymbol{F}_{e_i}^i\right)$$

abides by the three conditions in the above theorem, so that there exists a unique categorical differential $(n+1)$ *-form* ω' *with values in* $\mathcal{E}$ *with*

$$\tilde{\omega}'(-, e_1, \cdots, e_{n+1}) = e_1 \cdots e_{n+1} \omega'(-)$$

The corollary allows of exterior differentiation.

5. The Categorical Ambrose-Palais-Singer Theorem

A *categorical connection* on $\mathcal{M}$ is a functor

$$\nabla : \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \to \mathcal{M}^{D^2}$$

abiding by the following conditions:

(1) The diagram

$$\begin{tikzcd} \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \arrow[r, "\nabla"] \arrow[d, "\pi_1"'] & \mathcal{M}^{D^2} \arrow[dl, "\boldsymbol{F}_0^2"] \\ \mathcal{M}^D & \end{tikzcd}$$

is commutative, where $\pi_1 : \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \to \mathcal{M}^D$ is the forgetful functor to the first $\mathcal{M}^D$.

(2) The diagram

$$\begin{tikzcd} \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \arrow[r, "\nabla"] \arrow[d, "\pi_2"'] & \mathcal{M}^{D^2} \arrow[dl, "\boldsymbol{F}_0^1"] \\ \mathcal{M}^D & \end{tikzcd}$$

is commutative, where $\pi_2 : \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \to \mathcal{M}^D$ is the forgetful functor to the second $\mathcal{M}^D$.

(3) For any $r \in \boldsymbol{R}$, the diagram

$$\begin{tikzcd} \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \arrow[r, "\boldsymbol{D}_r^1 \times_{\mathcal{M}} \mathrm{Id}_{\mathcal{M}^D}"] \arrow[d, "\nabla"'] & \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \arrow[d, "\nabla"] \\ \mathcal{M}^D \arrow[r, "\boldsymbol{D}_r^1"'] & \mathcal{M}^D \end{tikzcd}$$

is commutative.

(4) For any $r \in \boldsymbol{R}$, the diagram

$$\begin{array}{ccc} \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D & \overset{\mathrm{Id}_{\mathcal{M}^D} \times \boldsymbol{D}_r^1}{\rightarrow} & \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \\ {}_{\nabla}\downarrow & & \downarrow_{\nabla} \\ \mathcal{M}^D & \underset{\boldsymbol{D}_r^2}{\rightarrow} & \mathcal{M}^D \end{array}$$

is commutative.

A categorical connection $\nabla : \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \to \mathcal{M}^{D^2}$ on $\mathcal{M}$ is called *symmetric* if the diagram

$$\begin{array}{ccc} \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \simeq \mathcal{M}^{D(2)} & \overset{\nabla}{\rightarrow} & \mathcal{M}^{D^2} \\ {}_{\boldsymbol{D}^\sigma}\downarrow & & \downarrow_{\boldsymbol{D}^\sigma} \\ \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \simeq \mathcal{M}^{D(2)} & \overset{\nabla}{\rightarrow} & \mathcal{M}^{D^2} \end{array}$$

is commutative, where σ is the permutation of 1 and 2.

A *categorical spray* on $\mathcal{M}$ is a functor $\Delta : \mathcal{M}^D \to \mathcal{M}^{D_2}$ abiding by the following conditions:

(1) The diagram

$$\begin{array}{ccc} \mathcal{M}^D & & \\ {}_{\Delta}\downarrow & \overset{\mathrm{Id}_{\mathcal{M}^D}}{\searrow} & \\ \mathcal{M}^{D_2} & \underset{\mathcal{M}^i}{\rightarrow} & \mathcal{M}^D \end{array}$$

is commutative, where $i : D \to D_2$ is the natural injection inducing the functor $\mathcal{M}^i : \mathcal{M}^{D_2} \to \mathcal{M}^D$.

(2) For any $r \in \boldsymbol{R}$, the diagram

$$\begin{array}{ccc} \mathcal{M}^D & \overset{\Delta}{\rightarrow} & \mathcal{M}^{D_2} \\ {}_{\boldsymbol{D}_r^1}\downarrow & & \downarrow_{\boldsymbol{D}_r^1} \\ \mathcal{M}^D & \underset{\Delta}{\rightarrow} & \mathcal{M}^{D_2} \end{array}$$

is commutative.

Proposition 5.1. *For a categorical connection* ∇ *on* $\mathcal{M}$, *there is a unique functor* Δ_∇: $\mathcal{M}^D \to \mathcal{M}^{D_2}$ *making the diagram commutativity of the diagram*

$$\begin{array}{ccc} \mathcal{M}^D & \overset{\Delta_\nabla}{\rightarrow} & \mathcal{M}^{D_2} \\ {}_{\left(\mathrm{Id}_{\mathcal{M}^D}, \mathrm{Id}_{\mathcal{M}^D}\right)}\downarrow & & \downarrow_{\mathcal{M}^{(d_1,d_2) \in D^2 \mapsto d_1+d_2 \in D_2}} \\ \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D & \overset{\nabla}{\rightarrow} & \mathcal{M}^{D^2} \end{array}$$

commutative. The functor Δ_∇ *is a categorical spray on* $\mathcal{M}$.

Proof. The crucial part of the proof is based on the following quasi-colimit diagram:

$$D \underset{d \in D \mapsto (0,d) \in D^2}{\overset{d \in D \mapsto (d,0) \in D^2}{\rightrightarrows}} D^2 \overset{(d_1,d_2) \in D^2 \mapsto d_1+d_2 \in D_2}{\rightarrow} D_2$$

owing to which, the unique functor $\Delta_\nabla : \mathcal{M}^D \to \mathcal{M}^{D_2}$ characterized by the commutativity of the above diagram exists, because the diagram

$$\mathcal{M}^D \xrightarrow{\left(\mathrm{Id}_{\mathcal{M}^D},\mathrm{Id}_{\mathcal{M}^D}\right)} \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \xrightarrow{\nabla} \mathcal{M}^{D^2} \underset{\mathcal{M}^{d\in D\mapsto (0,d)\in D^2}}{\overset{\mathcal{M}^{d\in D\mapsto (d,0)\in D^2}}{\rightrightarrows}} M^D$$

is commutative, as can be seen easily. The funcor $\Delta_\nabla : \mathcal{M}^D \to \mathcal{M}^{D_2}$ thus obtained is indeed a categorical spray on $\mathcal{M}$. $\square$

Conversely, a categorical spray on $\mathcal{M}$ determines a symmetric categorical connection on $\mathcal{M}$.

Proposition 5.2. *For a categorical spray* Δ *on* $\mathcal{M}$, *there is a unique functor* $\nabla_\Delta : \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \to \mathcal{M}^{D^2}$ *making the diagram give characterized by the commutativity of the diagram*

$$\begin{array}{ccc}
\mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D & \xrightarrow{\nabla_\Delta} & \mathcal{M}^{D^2} \\
{\scriptstyle (d_1,d_2)\in D^2 \mapsto \boldsymbol{D}^+ \circ \left(\boldsymbol{D}^{d_1} \times_{\mathcal{M}} \boldsymbol{D}^{d_2}\right)} \downarrow & & \downarrow {\scriptstyle e\in D_2 \mapsto \mathcal{M}^{(d_1,d_2)\in D^2 \mapsto (ed_1,ed_2)\in D^2}} \\
\mathcal{M}^{D\times D^2} \cong \left(\mathcal{M}^D\right)^{D^2} & \underset{\Delta^{D^2}}{\rightarrow} & \mathcal{M}^{D_2\times D^2} \cong \left(\mathcal{M}^{D^2}\right)^{D_2}
\end{array}$$

commutative. The functor ∇_Δ *is a symmetric categorical connection* ∇_Δ *on* $\mathcal{M}$.

Proof. The crucial part of the proof is based on the quasi-colimit diagram

$$\left(D_2\right)^2 \times D^2 \underset{(e_1,e_2,d_1,d_2)\in (D_2)^2\times D^2 \mapsto (e_2,e_1d_1,e_1d_2)\in D_2\times D^2}{\overset{(e_1,e_2,d_1,d_2)\in (D_2)^2\times D^2 \mapsto (e_1e_2,d_1,d_2)\in D_2\times D^2}{\rightrightarrows}} D_2 \times D^2 \xrightarrow{(e,d_1,d_2)\mapsto (ed_1,ed_2)} D^2$$

owing to which, there is a unique functor $\nabla_\Delta : \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \to \mathcal{M}^{D^2}$ making the above diagram commutative, because the diagram

$$\mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \xrightarrow{(d_1,d_2)\in D^2 \mapsto \boldsymbol{D}^+ \circ \left(\boldsymbol{D}^{d_1} \times_{\mathcal{M}} \boldsymbol{D}^{d_2}\right)} \mathcal{M}^{D\times D^2} \xrightarrow{\Delta^{D^2}} \mathcal{M}^{D_2\times D^2}$$

$$\underset{\mathcal{M}^{(e_1,e_2,d_1,d_2)\in (D_2)^2\times D^2 \mapsto (e_2,e_1d_1,e_1d_2)\in D_2\times D^2}}{\overset{\mathcal{M}^{(e_1,e_2,d_1,d_2)\in (D_2)^2\times D^2 \mapsto (e_1e_2,d_1,d_2)\in D_2\times D^2}}{\rightrightarrows}} \mathcal{M}^{(D_2)^2\times D^2}$$

is commutative. It is easy to see that the functor $\nabla_\Delta : \mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D \to \mathcal{M}^{D^2}$ thus obtained is indeed a symmetric canonical connection on $\mathcal{M}$.

Theorem 5.3. *For any categorical spray* Δ *on* $\mathcal{M}$, *we have*

$$\Delta = \Delta_{\nabla_\Delta}$$

Proof. We will show that, for any symmetric categorical connection ∇ on $\mathcal{M}$, there is a unique functor $\overline{\Delta_{\nabla_\Delta}} : \mathcal{M}^D \to \mathcal{M}^{D_2}$ making the diagram

$$\begin{array}{ccc}
\mathcal{M}^D & \xrightarrow{\overline{\Delta_{\nabla_\Delta}}} & \mathcal{M}^{D_2} \\
{\scriptstyle \left(\mathrm{Id}_{\mathcal{M}^D},\mathrm{Id}_{\mathcal{M}^D}\right)} \downarrow & & \downarrow {\scriptstyle \mathcal{M}^{(d_1,d_2)\in D^2 \mapsto d_1+d_2 \in D_2}} \\
\mathcal{M}^D \times_{\mathcal{M}} \mathcal{M}^D & & \mathcal{M}^{D^2} \\
{\scriptstyle (d_1,d_2)\in D^2 \mapsto \boldsymbol{D}^+ \circ \left(\boldsymbol{D}^{d_1} \times_{\mathcal{M}} \boldsymbol{D}^{d_2}\right)} \downarrow & & \downarrow {\scriptstyle e\in D_2 \mapsto \mathcal{M}^{(d_1,d_2)\in D^2 \mapsto (ed_1,ed_2)\in D^2}} \\
\mathcal{M}^{D\times D^2} \cong \left(\mathcal{M}^D\right)^{D^2} & \underset{\Delta^{D^2}}{\rightarrow} & \mathcal{M}^{D_2\times D^2} \cong \left(\mathcal{M}^{D^2}\right)^{D_2}
\end{array}$$

commutative. We simply juxtapose the commutative diagram characterizing ∇_Δ

and the commutative diagram characterizing to get the commutative diagram

$$\begin{array}{ccc}
\mathcal{M}^{D} & \overset{\Delta_{\nabla_\Delta}}{\rightarrow} & \mathcal{M}^{D_2} \\
\downarrow_{\left(\mathrm{Id}_{\mathcal{M}^D},\mathrm{Id}_{\mathcal{M}^D}\right)} & & \downarrow_{\mathcal{M}^{(d_1,d_2)\in D^2\mapsto d_1+d_2\in D_2}} \\
\mathcal{M}^{D}\times_{\mathcal{M}}\mathcal{M}^{D} & \overset{\nabla_\Delta}{\rightarrow} & \mathcal{M}^{D^2} \\
\downarrow_{(d_1,d_2)\in D^2\mapsto \boldsymbol{D}^{+}\circ\left(\boldsymbol{D}^{d_1}\times_{\mathcal{M}}\boldsymbol{D}^{d_2}\right)} & & \downarrow_{e\in D_2\mapsto \mathcal{M}^{(d_1,d_2)\in D^2\mapsto (ed_1,ed_2)\in D^2}} \\
\mathcal{M}^{D\times D^2}\cong\left(\mathcal{M}^{D}\right)^{D^2} & \underset{\Delta^{D^2}}{\rightarrow} & \mathcal{M}^{D_2\times D^2}\cong\left(\mathcal{M}^{D^2}\right)^{D_2}
\end{array}$$

which means that Δ_{∇_Δ} plays the role of $\overline{\Delta_{\nabla_\Delta}}$. Since the functor $e\in D_2\mapsto \mathcal{M}^{(d_1,d_2)\in D^2\mapsto (ed_1,ed_2)\in D^2}$ from $\mathcal{M}^{D^2}$ to $\mathcal{M}^{D_2\times D^2}$ is a monomorphism in $\boldsymbol{Cat}$, we can see easily that the commutativity of the outer square in the diagram

$$\begin{array}{ccc}
\mathcal{M}^{D} & \overset{\overline{\Delta_{\nabla_\Delta}}}{\rightarrow} & \mathcal{M}^{D_2} \\
\left(\mathrm{Id}_{\mathcal{M}^D},\mathrm{Id}_{\mathcal{M}^D}\right)\downarrow & & \downarrow_{\mathcal{M}^{(d_1,d_2)\in D^2\mapsto d_1+d_2\in D_2}} \\
\mathcal{M}^{D}\times_{\mathcal{M}}\mathcal{M}^{D} & \overset{\nabla_\Delta}{\rightarrow} & \mathcal{M}^{D^2} \\
(d_1,d_2)\in D^2\mapsto \boldsymbol{D}^{+}\circ\left(\boldsymbol{D}^{d_1}\times_{\mathcal{M}}\boldsymbol{D}^{d_2}\right)\downarrow & & \downarrow_{e\in D_2\mapsto \mathcal{M}^{(d_1,d_2)\in D^2\mapsto (ed_1,ed_2)\in D^2}} \\
\mathcal{M}^{D\times D^2}\cong\left(\mathcal{M}^{D}\right)^{D^2} & \underset{\Delta^{D^2}}{\rightarrow} & \mathcal{M}^{D_2\times D^2}\cong\left(\mathcal{M}^{D^2}\right)^{D_2}
\end{array}$$

is equivalent to the commutativity of the upper square, which characterizes the functor Δ_{∇_Δ}. Owing to Proposition 5.1, this means the unique existence of $\overline{\Delta_{\nabla_\Delta}}$ so that we have

$$\overline{\Delta_{\nabla_\Delta}}=\Delta_{\nabla_\Delta}$$

However, it is easy to that the diagram

$$\begin{array}{ccc}
\mathcal{M}^{D} & \overset{\Delta}{\rightarrow} & \mathcal{M}^{D_2} \\
\left(\mathrm{Id}_{\mathcal{M}^D},\mathrm{Id}_{\mathcal{M}^D}\right)\downarrow & & \downarrow_{\mathcal{M}^{(d_1,d_2)\in D^2\mapsto d_1+d_2\in D_2}} \\
\mathcal{M}^{D}\times_{\mathcal{M}}\mathcal{M}^{D} & & \mathcal{M}^{D^2} \\
(d_1,d_2)\in D^2\mapsto \boldsymbol{D}^{+}\circ\left(\boldsymbol{D}^{d_1}\times_{\mathcal{M}}\boldsymbol{D}^{d_2}\right)\downarrow & & \downarrow_{e\in D_2\mapsto \mathcal{M}^{(d_1,d_2)\in D^2\mapsto (ed_1,ed_2)\in D^2}} \\
\mathcal{M}^{D\times D^2}\cong\left(\mathcal{M}^{D}\right)^{D^2} & \underset{\Delta^{D^2}}{\rightarrow} & \mathcal{M}^{D_2\times D^2}\cong\left(\mathcal{M}^{D^2}\right)^{D_2}
\end{array}$$

is commutative, so that we have

$$\Delta=\Delta_{\nabla_\Delta}$$

□

Theorem 5.4. *For any symmetric categorical connection* ∇ *on* $\mathcal{M}$, *we have*

$$\nabla=\nabla_{\Delta_\nabla}$$

Proof. We will show that, for any symmetric categorical connection ∇ on $\mathcal{M}$, there is a unique functor $\overline{\nabla_{\Delta_\nabla}}:\mathcal{M}^{D}\times_{\mathcal{M}}\mathcal{M}^{D}\rightarrow\mathcal{M}^{D^2}$ making the diagram

$$\begin{array}{ccc}
\mathcal{M}^D\times_{\mathcal{M}}\mathcal{M}^D & \overset{\overline{\nabla_{\Delta_\nabla}}}{\rightarrow} & \mathcal{M}^{D^2} \\
{\scriptstyle (d_1,d_2)\in D^2\mapsto \boldsymbol{D}^+\circ\left(\boldsymbol{D}^{d_1}\times_{\mathcal{M}}\boldsymbol{D}^{d_2}\right)}\downarrow & & \downarrow{\scriptstyle e\in D_2\mapsto \mathcal{M}^{(d_1,d_2)\in D^2\mapsto(ed_1,ed_2)\in D^2}} \\
\mathcal{M}^{D\times D^2} & & \mathcal{M}^{D_2\times D^2} \\
{\scriptstyle \left(\mathrm{Id}_{\mathcal{M}^D},\mathrm{Id}_{\mathcal{M}^D}\right)^{D^2}}\downarrow & & \downarrow{\scriptstyle \mathcal{M}^{\left((d_1,d_2)\in D^2\mapsto d_1+d_2\in D_2\right)^{D^2}}} \\
\left(\mathcal{M}^D\times_{\mathcal{M}}\mathcal{M}^D\right)^{D^2} & \underset{\nabla^{D^2}}{\rightarrow} & \mathcal{M}^{D^2\times D^2}
\end{array}$$

commutative. We simply juxtapose the commutative diagram characterizing $\Delta_\nabla:\mathcal{M}^D\to\mathcal{M}^{D_2}$ after exponentiation with D^2 and the commutative diagram characterizing to get the commutative diagram

$$\begin{array}{ccc}
\mathcal{M}^D\times_{\mathcal{M}}\mathcal{M}^D & \overset{\nabla_{\Delta_\nabla}}{\rightarrow} & \mathcal{M}^{D^2} \\
{\scriptstyle (d_1,d_2)\in D^2\mapsto \boldsymbol{D}^+\circ\left(\boldsymbol{D}^{d_1}\times_{\mathcal{M}}\boldsymbol{D}^{d_2}\right)}\downarrow & & \downarrow{\scriptstyle e\in D_2\mapsto \mathcal{M}^{(d_1,d_2)\in D^2\mapsto(ed_1,ed_2)\in D^2}} \\
\mathcal{M}^{D\times D^2} & \overset{(\Delta_\nabla)^{D^2}}{\rightarrow} & \mathcal{M}^{D_2\times D^2} \\
{\scriptstyle \left(\mathrm{Id}_{\mathcal{M}^D},\mathrm{Id}_{\mathcal{M}^D}\right)^{D^2}}\downarrow & & \downarrow{\scriptstyle \mathcal{M}^{\left((d_1,d_2)\in D^2\mapsto d_1+d_2\in D_2\right)^{D^2}}} \\
\left(\mathcal{M}^D\times_{\mathcal{M}}\mathcal{M}^D\right)^{D^2} & \underset{\nabla^{D^2}}{\rightarrow} & \mathcal{M}^{D^2\times D^2}
\end{array}$$

which means that the functor ∇_{Δ_∇} plays the role of $\overline{\nabla_{\Delta_\nabla}}$. Considering the diagram

$$\begin{array}{ccc}
\mathcal{M}^D\times_{\mathcal{M}}\mathcal{M}^D & \overset{\overline{\nabla_{\Delta_\nabla}}}{\rightarrow} & \mathcal{M}^{D^2} \\
{\scriptstyle (d_1,d_2)\in D^2\mapsto \boldsymbol{D}^+\circ\left(\boldsymbol{D}^{d_1}\times_{\mathcal{M}}\boldsymbol{D}^{d_2}\right)}\downarrow & & \downarrow{\scriptstyle e\in D_2\mapsto \mathcal{M}^{(d_1,d_2)\in D^2\mapsto(ed_1,ed_2)\in D^2}} \\
\mathcal{M}^{D\times D^2} & \overset{(\Delta_\nabla)^{D^2}}{\rightarrow} & \mathcal{M}^{D_2\times D^2} \\
{\scriptstyle \left(\mathrm{Id}_{\mathcal{M}^D},\mathrm{Id}_{\mathcal{M}^D}\right)^{D^2}}\downarrow & & \downarrow{\scriptstyle \mathcal{M}^{\left((d_1,d_2)\in D^2\mapsto d_1+d_2\in D_2\right)^{D^2}}} \\
\left(\mathcal{M}^D\times_{\mathcal{M}}\mathcal{M}^D\right)^{D^2} & \underset{\nabla^{D^2}}{\rightarrow} & \mathcal{M}^{D^2\times D^2}
\end{array}$$

and taking into account the fact that the functor $\mathcal{M}^{\left((d_1,d_2)\in D^2\mapsto d_1+d_2\in D_2\right)^{D^2}}$ is monomorphism in the category ***Cat***, we can see that the outer square is commutative if the upper square is commutative. Therefore, we are sure of the unique existence of the desired functor $\overline{\nabla_{\Delta_\nabla}}$ by Proposition 5.2. It is also easy to see that the diagram

$$\begin{array}{ccc}
\mathcal{M}^D\times_{\mathcal{M}}\mathcal{M}^D & \overset{\nabla}{\rightarrow} & \mathcal{M}^{D^2} \\
{\scriptstyle (d_1,d_2)\in D^2\mapsto \boldsymbol{D}^+\circ\left(\boldsymbol{D}^{d_1}\times_{\mathcal{M}}\boldsymbol{D}^{d_2}\right)}\downarrow & & \downarrow{\scriptstyle e\in D_2\mapsto \mathcal{M}^{(d_1,d_2)\in D^2\mapsto(ed_1,ed_2)\in D^2}} \\
\mathcal{M}^{D\times D^2} & & \mathcal{M}^{D_2\times D^2} \\
{\scriptstyle \left(\mathrm{Id}_{\mathcal{M}^D},\mathrm{Id}_{\mathcal{M}^D}\right)^{D^2}}\downarrow & & \downarrow{\scriptstyle \mathcal{M}^{\left((d_1,d_2)\in D^2\mapsto d_1+d_2\in D_2\right)^{D^2}}} \\
\left(\mathcal{M}^D\times_{\mathcal{M}}\mathcal{M}^D\right)^{D^2} & \underset{\nabla^{D^2}}{\rightarrow} & \mathcal{M}^{D^2\times D^2}
\end{array}$$

is commutative, so that we have

$$\nabla = \nabla_{\Delta_{\nabla}}$$

□

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix. The Proof of Theorem 3.7

Appendix A. Bilinearity

The following simple proposition can be proved as in classical SDG (Lemma and Proposition 10, [2]).

Proposition A.1. *Let* F *be a functor from a categorical* $\mathbf{R}$*-module* $\mathcal{V}$ *to a Euclidean categorical* $\mathbf{R}$*-module* $\mathcal{E}$*. If* F *is homogeneous, then* F *is linear.*

By this proposition, it suffices to establish the bi-homogeneity, which follows directly from the definition of the Lie bracket.

Appendix B. Strong Differences

In the following, we are concerned with antisymmetry and the Jacobi identity. The discussion is an elaboration of our previous discussion [6]-[8].

The notion of *strong difference* $\dot{-}$ is discussed in orthodox differential geometry. Given two microsquares γ_1 and γ_2 on a microlinear space $\mathcal{M}$ with $\gamma_1|_{D(2)} = \gamma_2|_{D(2)}$, their strong difference $\gamma_2 \dot{-} \gamma_1$ is defined on account of the following quasi-colimit diagram

$$
\begin{tikzcd}
D(2) \arrow[r, "(d_1,d_2)\in D(2)\mapsto(d_1,d_2)\in D^2"] \arrow[d, "(d_1,d_2)\in D(2)\mapsto\left(d_1,d_2\in D^2\right)"'] & D^2 \arrow[d, "(d_1,d_2)\in D^2\mapsto(d_1,d_2,d_1d_2)\in D^2\oplus D"] \\
D^2 \arrow[r, "(d_1,d_2)\in D^2\mapsto(d_1,d_2,0)\in D^2\oplus D"'] & D^2\oplus D
\end{tikzcd}
$$

We have the following theorem, whose proof based on a quasi-colimit diagram is given in the next subsection.

Theorem B.1. *For any microsquares* γ_{12} *and* γ_{21} *on a microlinear space* $\mathcal{M}$ *with* $\gamma_{12}|_{D(2)} = \gamma_{21}|_{D(2)}$*, we have*

$$\gamma_{12} - \dot{-} \gamma_{21}$$

$$\gamma_{21} - \dot{-} \gamma_{12}$$

which sum up only to vanish.

The notion of strong differences can be relativised, resulting in *relative strong differences* $\underset{1}{\dot{-}}$, $\underset{2}{\dot{-}}$ and $\underset{3}{\dot{-}}$ between microcubes on a microlinear space $\mathcal{M}$. By identifying a microcube γ on $\mathcal{M}$ with a microsquare on $\mathcal{M}^D$

$$(d_2,d_3)\in D^2 \mapsto \left(d_1\in D \mapsto \gamma(d_1,d_2,d_3)\in\mathcal{M}\right)$$

the notion of strong difference $-\dot{-}$ between microsquares on $\mathcal{M}^D$ becomes the notion of relative strong difference $\underset{1}{\dot{-}}$ between microcubes on $\mathcal{M}$. Given microcubes γ and ζ on $\mathcal{M}$ with

$$\gamma|_{(D\times D\times D_0)\cup(D\times D_0\times D)} = \zeta|_{(D\times D\times D_0)\cup(D\times D_0\times D)}$$

the relative strong difference $\zeta \underset{1}{\dot{-}} \gamma$ is defined to be a microsquare on $\mathcal{M}$. By identifying a microcube γ on $\mathcal{M}$ with a microsquare on $\mathcal{M}^D$

$$\left(d_1,d_3\right)\in D^2 \mapsto \left(d_2 \in D \mapsto \gamma\left(d_1,d_2,d_3\right)\in\mathcal{M}\right)$$

the notion of strong difference $\overset{\cdot}{-}$ between microsquares on $\mathcal{M}^D$ becomes the notion of relative strong difference $\underset{2}{\overset{\cdot}{-}}$ between microcubes on $\mathcal{M}$. Given microcubes γ and ζ on $\mathcal{M}$ with

$$\gamma|_{(D\times D\times D_0)\cup(D_0\times D\times D)} = \zeta|_{(D\times D\times D_0)\cup(D_0\times D\times D)}$$

the relative strong difference $\zeta \underset{2}{\overset{\cdot}{-}} \gamma$ is defined to be a microsquare on $\mathcal{M}$. By identifying a microcube γ on $\mathcal{M}$ with a microsquare on $\mathcal{M}^D$

$$\left(d_1,d_2\right)\in D^2 \mapsto \left(d_3 \in D \mapsto \gamma\left(d_1,d_2,d_3\right)\in\mathcal{M}\right)$$

the notion of strong difference $\overset{\cdot}{--}$ between microsquares on $\mathcal{M}^D$ becomes the notion of relative strong difference $\underset{3}{\overset{\cdot}{-}}$ between microcubes on $\mathcal{M}$. Given microcubes γ and ζ on $\mathcal{M}$ with

$$\gamma|_{(D\times D_0\times D)\cup(D_0\times D\times D)} = \zeta|_{(D\times D_0\times D)\cup(D_0\times D\times D)}$$

the relative strong difference $\zeta \underset{3}{\overset{\cdot}{-}} \gamma$ is defined to be a microsquare on $\mathcal{M}$.

Lemma B.2. *We have the following*:

(1) *Given microcubes* γ, ζ, γ' *and* ζ' *with* $\gamma|_{(D\times D\times D_0)\cup(D\times D_0\times D)} = \zeta|_{(D\times D\times D_0)\cup(D\times D_0\times D)}$ *and* $\gamma'|_{(D\times D\times D_0)\cup(D\times D_0\times D)} = \zeta'|_{(D\times D\times D_0)\cup(D\times D_0\times D)}$, *we have* $\zeta \underset{1}{\overset{\cdot}{-}} \gamma$ *and* $\zeta' \underset{1}{\overset{\cdot}{-}} \gamma'$. *If they satisfy* $\gamma|_{D\oplus D^2} = \gamma'|_{D\oplus D^2}$ *and* $\zeta|_{D\oplus D^2} = \zeta'|_{D\oplus D^2}$, *then we have*

$$\left(\zeta \underset{1}{\overset{\cdot}{-}} \gamma\right)\Big|_{D(2)} = \left(\zeta' \underset{1}{\overset{\cdot}{-}} \gamma'\right)\Big|_{D(2)} \text{ so that } \left(\zeta' \underset{1}{\overset{\cdot}{-}} \gamma'\right) \overset{\cdot}{-} \left(\zeta \underset{1}{\overset{\cdot}{-}} \gamma\right) \text{ is defined.}$$

(2) *Given microcubes* γ, ζ, γ' *and* ζ' *with* $\gamma|_{(D\times D\times D_0)\cup(D_0\times D\times D)} = \zeta|_{(D\times D\times D_0)\cup(D_0\times D\times D)}$ *and* $\gamma'|_{(D\times D\times D_0)\cup(D_0\times D\times D)} = \zeta'|_{(D\times D\times D_0)\cup(D_0\times D\times D)}$, *we have* $\zeta \underset{2}{\overset{\cdot}{-}} \gamma$ *and* $\zeta' \underset{2}{\overset{\cdot}{-}} \gamma'$. *If they satisfy* $\gamma \circ \left(\left(d_1,d_2,d_3\right)\in D\oplus D^2 \mapsto \left(d_1,d_2,d_3\right)\in D^3\right) = \gamma' \circ \left(\left(d_1,d_2,d_3\right)\in D\oplus D^2 \mapsto \left(d_1,d_2,d_3\right)\in D^3\right)$ *and* $\zeta \circ \left(\left(d_1,d_2,d_3\right)\in D\oplus D^2 \mapsto \left(d_1,d_2,d_3\right)\in D^3\right) = \zeta' \circ \left(\left(d_1,d_2,d_3\right)\in D\oplus D^2 \mapsto \left(d_1,d_2,d_3\right)\in D^3\right)$, *then we have*

$$\left(\zeta \underset{2}{\overset{\cdot}{-}} \gamma\right)\Big|_{D(2)} = \left(\zeta' \underset{2}{\overset{\cdot}{-}} \gamma'\right)\Big|_{D(2)} \text{ so that } \left(\zeta' \underset{2}{\overset{\cdot}{-}} \gamma'\right) \overset{\cdot}{-} \left(\zeta \underset{2}{\overset{\cdot}{-}} \gamma\right) \text{ is defined.}$$

(3) *Given microcubes* γ, ζ, γ' *and* ζ' *with* $\gamma|_{(D\times D_0\times D)\cup(D_0\times D\times D)} = \zeta|_{(D\times D_0\times D)\cup(D_0\times D\times D)}$ *and* $\gamma'|_{(D\times D_0\times D)\cup(D_0\times D\times D)} = \zeta'|_{(D\times D_0\times D)\cup(D_0\times D\times D)}$, *we have* $\zeta \underset{3}{\overset{\cdot}{-}} \gamma$ *and* $\zeta' \underset{3}{\overset{\cdot}{-}} \gamma'$. *If they sat-*

isfy $\gamma|_{D^2\oplus D} = \gamma'|_{D^2\oplus D}$ *and* $\zeta|_{D^2\oplus D} = \zeta'|_{D^2\oplus D}$, *then we have*

$\left(\zeta \underset{3}{\dot{-}} \gamma\right)\Big|_{D(2)} = \left(\zeta' \underset{3}{\dot{-}} \gamma'\right)\Big|_{D(2)}$ *so that* $\left(\zeta' \underset{3}{\dot{-}} \gamma'\right) \dot{-} \left(\zeta \underset{3}{\dot{-}} \gamma\right)$ *is defined.*

Now we have the following theorem, which is the 3-dimensional generalization of the 2-dimensional general antisymmetry.

Theorem B.3. *Let* γ_{123}, γ_{132}, γ_{213}, γ_{231}, γ_{312} *and* γ_{321} *be microcubes on a microlinear space* $\mathcal{M}$ *making the following diagram commutative, where the diagram consists of the following vertices*

$$\begin{array}{cccccc}
\underset{[13]}{D^3} & \underset{[14]}{D^3} & \underset{[15]}{D^3} & \underset{[16]}{D^3} & \underset{[17]}{D^3} & \underset{[18]}{D^3} \\
(\gamma_{123}) & (\gamma_{132}) & (\gamma_{213}) & (\gamma_{231}) & (\gamma_{312}) & (\gamma_{321}) \\
\underset{[7]}{D^2 \oplus D^2} & \underset{[8]}{D^2 \oplus D^2} & \underset{[9]}{D^2 \oplus D^2} & \underset{[10]}{D^2 \oplus D^2} & \underset{[11]}{D^2 \oplus D^2} & \underset{[12]}{D^2 \oplus D^2} \\
\underset{[1]}{D^2 \oplus D} & \underset{[2]}{D^2 \oplus D} & \underset{[3]}{D^2 \oplus D} & \underset{[4]}{D^2 \oplus D} & \underset{[5]}{D^2 \oplus D} & \underset{[6]}{D^2 \oplus D}
\end{array}$$

together with $\mathcal{M}$ *as well as the following mappings:*

(1) *The mappings* $i_{[1][13]} : [1] \to [13]$, $i_{[2][14]}$, $i_{[3][16]}$, $i_{[4][15]}$, $i_{[5][17]}$ *and* $i_{[6][18]}$ *are*

$$(d_1, d_2, e) \in D^2 \oplus D \mapsto (e, d_1, d_2) \in D^3$$

(2) *The mappings* $i_{[1][16]}$ $i_{[2][18]}$, $i_{[3][17]}$, $i_{[4][14]}$, $i_{[5][13]}$ *and* $i_{[6][15]}$ *are*

$$(d_1, d_2, e) \in D^2 \oplus D \mapsto (d_1, d_2, e) \in D^3$$

(3) *The mappings* $i_{[7][13]}$, $i_{[7][14]}$, $i_{[8][16]}$, $i_{[8][15]}$, $i_{[9][17]}$ *and* $i_{[9][18]}$ *are*

$$(d_1, d_2, e_1, e_2) \in D^2 \oplus D^2 \mapsto (d_1 + e_1, d_2, e_2) \in D^3$$

(4) *The mappings* $\gamma_{123} : D^3 \to \mathcal{M}$, $\gamma_{132} : D^3 \to \mathcal{M}$, $\gamma_{213} : D^3 \to \mathcal{M}$, $\gamma_{231} : D^3 \to \mathcal{M}$, $\gamma_{312} : D^3 \to \mathcal{M}$ *and* $\gamma_{321} : D^3 \to \mathcal{M}$.

Then the following three expressions are meaningful, summing up only to vanish.

$$\left(\gamma_{123} \underset{1}{\dot{-}} \gamma_{132}\right) \dot{-} \left(\gamma_{231} \underset{1}{\dot{-}} \gamma_{321}\right)$$

$$\left(\gamma_{231} \underset{2}{\dot{-}} \gamma_{213}\right) \dot{-} \left(\gamma_{312} \underset{2}{\dot{-}} \gamma_{132}\right)$$

$$\left(\gamma_{312} \underset{3}{\dot{-}} \gamma_{321}\right) \dot{-} \left(\gamma_{123} \underset{3}{\dot{-}} \gamma_{213}\right)$$

The Lie bracket $[\cdot,\cdot]$ in $\left(\mathcal{M}^{\mathcal{M}}\right)^D_{\mathrm{Id}}$ can be expressed in terms of strong differences. Given $X, Y, Z \in \left(\mathcal{M}^{\mathcal{M}}\right)^D_{\mathrm{Id}}$, microsquares γ_{12} and γ_{21} on $\mathcal{M}^{\mathcal{M}}$ as well as microcubes γ_{123}, γ_{132}, γ_{213}, γ_{231}, γ_{312} and γ_{321} on $\mathcal{M}^{\mathcal{M}}$ as follows:

$$\gamma_{12}(d_1, d_2) = Y_{d_2} \circ X_{d_1}$$

$$\gamma_{21}(d_1, d_2) = X_{d_1} \circ Y_{d_2}$$

$$\gamma_{123}(d_1, d_2, d_3) = Z_{d_3} \circ Y_{d_2} \circ X_{d_1}$$

$$\gamma_{132}\left(d_1,d_2,d_3\right)=Y_{d_2}\circ Z_{d_3}\circ X_{d_1}$$

$$\gamma_{213}\left(d_1,d_2,d_3\right)=Z_{d_3}\circ X_{d_1}\circ Y_{d_2}$$

$$\gamma_{231}\left(d_1,d_2,d_3\right)=X_{d_1}\circ Z_{d_3}\circ Y_{d_2}$$

$$\gamma_{312}\left(d_1,d_2,d_3\right)=Y_{d_2}\circ X_{d_1}\circ Z_{d_3}$$

$$\gamma_{321}\left(d_1,d_2,d_3\right)=X_{d_1}\circ Y_{d_2}\circ Z_{d_3}$$

Lemma B.4. *We have*

$$\left[X,Y\right]=\gamma_{12}\overset{\cdot}{--}\gamma_{21}$$

$$\left[Y,X\right]=\gamma_{21}\overset{\cdot}{--}\gamma_{12}$$

$$\left[X,\left[Y,Z\right]\right]=\left(\gamma_{123}\underset{1}{\overset{\cdot}{-}}\gamma_{132}\right)\overset{\cdot}{-}\left(\gamma_{231}\underset{1}{\overset{\cdot}{-}}\gamma_{321}\right)$$

$$\left[Y,\left[Z,X\right]\right]=\left(\gamma_{231}\underset{2}{\overset{\cdot}{-}}\gamma_{213}\right)\overset{\cdot}{-}\left(\gamma_{312}\underset{2}{\overset{\cdot}{-}}\gamma_{132}\right)$$

$$\left[Z,\left[X,Y\right]\right]=\left(\gamma_{312}\underset{3}{\overset{\cdot}{-}}\gamma_{321}\right)\overset{\cdot}{-}\left(\gamma_{123}\underset{3}{\overset{\cdot}{-}}\gamma_{213}\right)$$

Therefore, antisymmetry and the Jacobi identity of the Lie bracket follows directly from the general antisymmetry and the general Jacobi identity.

The discussion for microlinear spaces so far can easily be elevated to microlinear categories. Given a microlinear category $\mathcal{M}$, we define a functor $\boldsymbol{D}^{\overset{\cdot}{-}}:\mathcal{M}^{D^2}\times_{D(2)}\mathcal{M}^{D^2}\to\mathcal{M}^{D}$ in place of the mapping $\overset{\cdot}{-}$, $\cdots$.

Appendix C. General Antisymmetry

The general antisymmetry follows from the quasi-colimit diagram whose vertices go as follows:

$$\begin{array}{ccc}
 & \underset{[7]}{D^2\oplus D_0} & \\
 & \underset{[6]}{D^2\oplus D(2)} & \\
\underset{[4]}{D^2\oplus D} & \underset{[8]}{D^2\oplus D} & \underset{[5]}{D^2\oplus D} \\
\left(\gamma_{12}\overset{\cdot}{--}\gamma_{21}\right) & & \left(\gamma_{21}\overset{\cdot}{--}\gamma_{12}\right) \\
\underset{[2]}{D^2} & & \underset{[3]}{D^2} \\
\left(\gamma_{12}\right) & & \left(\gamma_{21}\right) \\
 & \underset{[1]}{D(2)} &
\end{array}$$

Its arrows go as follows:

(1) The mappings $i_{[1][2]}$ from $[1]$ to $[2]$ and $i_{[1][3]}$ from $[1]$ to $[3]$ are

$$\left(d_1,d_2\right)\in D(2)\mapsto\left(d_1,d_2\right)\in D^2$$

(2) The mappings $i_{[2][4]}$ and $i_{[3][5]}$ are

$$\left(d_1,d_2\right)\in D^2\mapsto\left(d_1,d_2,0\right)\in D^2\oplus D$$

(3) The mappings $i_{[3][4]}$ and $i_{[2][5]}$ are

$$\left(d_1,d_2\right)\in D^2\mapsto\left(d_1,d_2,d_1d_2\right)\in D^2\oplus D$$

(4) The mapping $i_{[4][6]}$ is

$$\left(d_1,d_2,e\right)\in D^2\oplus D\mapsto\left(d_1,d_2,e,0\right)\in D^2\oplus D(2)$$

(5) The mapping $i_{[5][6]}$ is

$$\left(d_1,d_2,e\right)\in D^2\oplus D\mapsto\left(d_1,d_2,d_1d_2,e\right)\in D^2\oplus D(2)$$

(6) The mapping $i_{[6][7]}$ is

$$\left(d_1,d_2,e_1,e_2\right)\in D^2\oplus D(2)\mapsto\left(d_1,d_2,0\right)\in D^2\oplus D_0$$

(7) The mapping $i_{[8][6]}$ is

$$\left(d_1,d_2,e\right)\in D^2\oplus D\mapsto\left(d_1,d_2,e,e\right)\in D^2\oplus D(2)$$

Appendix D. General Jacobi Identity

The general Jacobi identity follows from the quasi-colimit diagram whose vertices go as follows:

$$\begin{array}{cccc}
 & & \underset{[29]}{D^3(3)\oplus D^2(3)\oplus D_0} & \\
 & & \underset{[28]}{D^3(3)\oplus D^2(3)\oplus D(3)} & \\
\underset{[30]}{D^3(3)\oplus D^2(3)\oplus D} & \underset{\substack{[25]\\ \left(\left(\gamma_{123}\overset{\cdot}{\underset{1}{-}}\gamma_{132}\right)\overset{\cdot}{-}\left(\gamma_{231}\overset{\cdot}{\underset{1}{-}}\gamma_{321}\right)\right)}}{D^3(2)\oplus D^2\oplus D} & \underset{\substack{[26]\\ \left(\gamma_{231}\overset{\cdot}{\underset{2}{-}}\gamma_{213}\right)\overset{\cdot}{-}\left(\gamma_{312}\overset{\cdot}{\underset{2}{-}}\gamma_{132}\right)}}{D^3(2)\oplus D^2\oplus D} & \underset{\substack{[27]\\ \left(\gamma_{312}\overset{\cdot}{\underset{3}{-}}\gamma_{321}\right)\overset{\cdot}{-}\left(\gamma_{123}\overset{\cdot}{\underset{3}{-}}\gamma_{213}\right)}}{D^3(2)\oplus D^2\oplus D} \\
\underset{\substack{[19]\\ \left(\gamma_{123}\overset{\cdot}{\underset{1}{-}}\gamma_{132}\right)}}{D^3\oplus D^2} & \underset{\substack{[20]\\ \left(\gamma_{231}\overset{\cdot}{\underset{2}{-}}\gamma_{213}\right)}}{D^3\oplus D^2} & \underset{\substack{[21]\\ \left(\gamma_{312}\overset{\cdot}{\underset{3}{-}}\gamma_{321}\right)}}{D^3\oplus D^2} & \underset{\substack{[22]\\ \left(\gamma_{231}\overset{\cdot}{\underset{1}{-}}\gamma_{321}\right)}}{D^3\oplus D^2} \\
\underset{\substack{[23]\\ \left(\gamma_{312}\overset{\cdot}{\underset{2}{-}}\gamma_{132}\right)}}{D^3\oplus D^2} & \underset{\substack{[24]\\ \left(\gamma_{123}\overset{\cdot}{\underset{3}{-}}\gamma_{213}\right)}}{D^3\oplus D^2} & & \\
\underset{\substack{[13]\\ (\gamma_{123})}}{D^3} & \underset{\substack{[14]\\ (\gamma_{132})}}{D^3} & \underset{\substack{[15]\\ (\gamma_{213})}}{D^3} & \underset{\substack{[16]\\ (\gamma_{231})}}{D^3} \\
\underset{\substack{[17]\\ (\gamma_{312})}}{D^3} & \underset{\substack{[18]\\ (\gamma_{321})}}{D^3} & & \\
\underset{[7]}{D^2\oplus D^2} & \underset{[8]}{D^2\oplus D^2} & \underset{[9]}{D^2\oplus D^2} & \underset{[10]}{D^2\oplus D^2} \\
\underset{[11]}{D^2\oplus D^2} & \underset{[12]}{D^2\oplus D^2} & & \\
\underset{[1]}{D^2\oplus D} & \underset{[2]}{D^2\oplus D} & \underset{[3]}{D^2\oplus D} & \underset{[4]}{D^2\oplus D} \\
\underset{[5]}{D^2\oplus D} & \underset{[6]}{D^2\oplus D} & &
\end{array}$$

Its arrows go as follows:

(1) The mappings $i_{[1][13]}$, $i_{[2][14]}$, $i_{[3][16]}$, $i_{[4][15]}$, $i_{[5][17]}$ and $i_{[6][18]}$ are

$$\left(d_1,d_2,e\right)\in D^2\oplus D\mapsto\left(e,d_1,d_2\right)\in D^3$$

(2) The mappings $i_{[1][16]}$, $i_{[2][18]}$, $i_{[3][17]}$, $i_{[4][14]}$, $i_{[5][13]}$ and $i_{[6][15]}$ are

$$\left(d_1,d_2,e\right)\in D^2\oplus D\mapsto\left(d_1,d_2,e\right)\in D^3$$

(3) The mappings $i_{[7][13]}$, $i_{[7][14]}$, $i_{[8][16]}$, $i_{[8][15]}$, $i_{[9][17]}$ and $i_{[9][18]}$ are

$$\left(d_1,d_2,e_1,e_2\right)\in D^2\oplus D^2\mapsto\left(d_1+e_1,d_2,e_2\right)\in D^3$$

(4) The mappings $i_{[10][16]}$, $i_{[10][18]}$, $i_{[11][17]}$, $i_{[11][14]}$, $i_{[12][13]}$ and $i_{[12][15]}$ are

$$\left(d_1,d_2,e_1,e_2\right)\in D^2\oplus D^2\mapsto\left(d_2,e_2,d_1+e_1\right)\in D^3$$

(5) The mappings $i_{[14][19]}$, $i_{[15][20]}$, $i_{[18][21]}$, $i_{[18][22]}$, $i_{[14][23]}$ and $i_{[15][24]}$ are

$$\left(d_1,d_2,d_3\right)\in D^3\mapsto\left(d_1,d_2,d_3,0,0\right)\in D^3\oplus D^2$$

(6) The mappings $i_{[13][19]}$, $i_{[16][20]}$ and $i_{[17][21]}$ are

$$\left(d_1,d_2,d_3\right)\in D^3\mapsto\left(d_1,d_2,d_3,d_1,d_2d_3\right)\in D^3\oplus D^2$$

(7) The mappings $i_{[16][22]}$, $i_{[17][23]}$ and $i_{[13][24]}$ are

$$\left(d_1,d_2,d_3\right)\in D^3\mapsto\left(d_1,d_2,d_3,d_3,d_1d_2\right)\in D^3\oplus D^2$$

(8) The mappings $i_{[22][25]}$, $i_{[23][26]}$ and $i_{[24][27]}$ are

$$\left(d_1,d_2,d_3,e_1,e_2\right)\in D^3\oplus D^2\mapsto\left(d_1,d_2,d_3,0,0,0,e_1,e_2,0\right)\in D^3\left(2\right)\oplus D^2\oplus D$$

(9) The mappings $i_{[19][25]}$, $i_{[20][26]}$ and $i_{[21][27]}$ are

$$\left(d_1,d_2,d_3,e_1,e_2\right)\in D^3\oplus D^2\mapsto\left(0,0,0,d_1,d_2,d_3,e_1,e_2,e_1e_2\right)\in D^3\left(2\right)\oplus D^2\oplus D$$

(10) The mapping $i_{[25][28]}$ is

$$\left(d_1,d_2,d_3,d_1',d_2',d_3',e_1,e_2,f\right)\in D^3\left(2\right)\oplus D^2\oplus D\mapsto$$

$$\left(d_1,d_2,d_3,d_1',d_2',d_3',0,0,0,e_1,e_2,0,0,0,0,f,0,0\right)\in D^3\left(3\right)\oplus D^2\left(3\right)\oplus D\left(3\right)$$

(11) The mapping $i_{[26][28]}$ is

$$\left(d_1,d_2,d_3,d_1',d_2',d_3',e_1,e_2,f\right)\in D^3\left(2\right)\oplus D^2\oplus D\mapsto$$

$$\left(0,0,0,d_1,d_2,d_3,d_1',d_2',d_3',0,0,e_1,e_2,0,0,0,f,0\right)\in D^3\left(3\right)\oplus D^2\left(3\right)\oplus D\left(3\right)$$

(12) The mapping $i_{[27][28]}$ is

$$\left(d_1,d_2,d_3,d_1',d_2',d_3',e_1,e_2,f\right)\in D^3\left(2\right)\oplus D^2\oplus D\mapsto$$

$$\left(d_1',d_2',d_3',0,0,0,d_1,d_2,d_3,0,0,0,0,e_1,e_2,0,0,f\right)\in D^3\left(3\right)\oplus D^2\left(3\right)\oplus D\left(3\right)$$

(13) The mapping $i_{[28][29]}$ is

$$\left(d_1,d_2,d_3,d_1',d_2',d_3',d_1'',d_2'',d_3'',e_1,e_2,e_1',e_2',e_1'',e_2'',f,f',f''\right)\in D^3\left(3\right)\oplus D^2\left(3\right)\oplus D\left(3\right)\mapsto$$

$$\left(d_1, d_2, d_3, d_1', d_2', d_3', d_1'', d_2'', d_3'', e_1, e_2, e_1', e_2', e_1'', e_2'', 0\right) \in D^3(3) \oplus D^2(3) \oplus D_0$$

(14) The mapping $i_{[30][28]}$ is

$$\left(d_1, d_2, d_3, d_1', d_2', d_3', d_1'', d_2'', d_3'', e_1, e_2, e_1', e_2', e_1'', e_2'', f\right) \in D^3(3) \oplus D^2(3) \oplus D \mapsto$$
$$\left(d_1, d_2, d_3, d_1', d_2', d_3', d_1'', d_2'', d_3'', e_1, e_2, e_1', e_2', e_1'', e_2'', f, f, f\right) \in D^3(3) \oplus D^2(3) \oplus D(3)$$