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The Well-Posedness of Forward-Backward Stochastic Partial Differential Equations

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Abstract

The solvability of a class of fully coupled forward-backward stochastic partial differential equations (FBSPDEs) under monotonicity conditions are studied in this paper. The existence and uniqueness of the solutions of the FBSPDEs have been obtained.

Keywords

Forward-Backward Equations, Navier-Stokes Equations, Four Step Scheme, Galerkin Approximation

1. Introduction

Stochastic differential equations (SDEs) are mathematical models used to describe the evolution of random phenomena in continuous time and space [1]. Forward-backward stochastic differential equations (FBSDEs) are a particular class of SDEs that involve both forward and backward stochastic processes. They arise naturally in a variety of fields, including finance, physics, and engineering [2]. Solving FBSDEs is a challenging task due to their inherent complexity and the need to handle both the forward and backward processes simultaneously. One approach to solving FBSDEs is the method of monotonicity, which has gained significant attention in recent years due to its ability to handle non-linearities and avoid numerical instability issues. The method of monotonicity was first introduce by Pardoux in his pioneering work [3] to solve stochastic partial differential equations, and later, further advancements were made by Krylov and Rozovskii [4], and Gyöngy [5]. The study of fully coupled FBSDEs under monotone type conditions were conducted in [6] [7], and favorable results are obtained. It was further developed in [8] [9].

Forward-backward stochastic partial differential equations (FBSPDEs) can be viewed as a natural extension of FBSDEs. In light of the nonlinear Feynman-Kac formula, or the Four Step Scheme [10], it is not hard to imagine that the solution of a backward SPDE could be a crucial device for solving an FBSDE with random coefficients [11] [12]. It has been shown that the solvability of a large class of non-Markovian FBSDEs is almost equivalent to the solvability of the corresponding backward stochastic partial differential equations (BSPDEs) [13]. Therefore the solvability of FBSPDEs could be considered as part of the effort for a full understanding of the solvability of general strongly coupled FBSDEs with random coefficients. In addition, the interesting structure of FBSPDEs can be used to describe many natural phenomena. For instance, an application to the reaction-diffusion models is provided in [14]. The solvability of FBSPDEs has been attempted using the method of contraction mapping and method of continuation under various of conditions and assumptions in [15]. In [16], Yosida approximation scheme has been employed to establish the solvability of FBSPDEs even in the absence of the Lipschitz conditions. However, this is accomplished by assuming non-degeneracy and a special compatibility condition to compensate for the lack of Lipschitz continuity. Under the method of Galerkin approximation, a weak form solution of FBSPDEs was obtained in [17].

The present paper is distinguished from [14]-[17] chiefly in the method employed and in the resulting class of couplings covered. In [15] well-posedness is obtained by contraction mapping and continuation under a Lipschitz framework, but with the forward and backward equations coupled through Lipschitz coefficients rather than through the sharper, one-sided monotonicity structure used here; [16] trades the Lipschitz assumption for a non-degeneracy and compatibility condition via Yosida approximation, a route that excludes the degenerate diffusion coefficients allowed under monotonicity; and [17] produces only a weak (Galerkin) solution, without the strong, adapted solution obtained below. Relative to all of [14]-[17], the present contribution is threefold: 1) we adapt the monotonicity method of Pardoux [3] and Krylov-Rozovskii [4] - previously used for single-direction (forward-only or backward-only) SPDEs - to the fully coupled forward-backward setting, which requires the new one-sided monotonicity assumptions (A.2)-(A.3) and (B.3) on the coupling terms themselves, rather than only Lipschitz bounds on their size; 2) this structure yields existence and uniqueness under weaker regularity on the coefficients b,σ,k than the Lipschitz-in-all-variables assumptions of [15], since only σ and the diagonal terms of b,k need be controlled via (A.2), while the full Lipschitz bound (A.3) is only needed for the coupling; and (iii) the abstract formulation (4.1) in Sections 4-5, obtained via a method of continuation on a coupling parameter $\alpha\in[0,1]$, subsumes both the Laplacian-type system (1.2) and the coercive-operator system (3.1) as special cases, broadening the class of admissible second-order (or more general continuous linear) operators beyond what the contraction arguments of [15] allow directly. In this paper, we would like to explore the method of monotonicity to

FBSPDEs. Let us first start with the following FBSPDEs.

$$\begin{cases} \partial_t \boldsymbol{u}(t,x) = a\sum_{i,j=1}^{d} \boldsymbol{u}_{x_i x_j}(t,x)\mathrm{d}t + b\left(t,x,\boldsymbol{u}(t,x),\boldsymbol{v}(t,x),Z(t,x)\right)\mathrm{d}t \\ \qquad\qquad + \sigma\left(t,x,\boldsymbol{u}(t,x),\boldsymbol{v}(t,x),Z(t,x)\right)\mathrm{d}W(t,x) \\ \partial_t \boldsymbol{v}(t,x) = -h\sum_{i,j=1}^{d} \boldsymbol{v}_{x_i x_j}(t,x)\mathrm{d}t + k\left(t,x,\boldsymbol{u}(t,x),\boldsymbol{v}(t,x),Z(t,x)\right)\mathrm{d}t \\ \qquad\qquad + Z(t,x)\mathrm{d}W(t,x) \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x) \text{ and } \boldsymbol{v}(T,x) = g\left(\boldsymbol{u}(T,x)\right), t\in[0,T], x\in G, \end{cases} \tag{1.1}$$

where a and h are positive constants, and G is a bounded domain in $\mathbb{R}^d$ with smooth boundary conditions. Let $|\cdot|$ be the norm of $L^2(G)$ and $\|\cdot\|$ be the norm of $H_0^1(G)$. They are given as follows:

$$|\boldsymbol{u}| \triangleq \left(\int_G |\boldsymbol{u}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}},$$

and

$$\|\boldsymbol{u}\| \triangleq \left(\int_G |\nabla \boldsymbol{u}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

For notational simplicity, the norm $|\cdot|$ inside the integral signs is also used to denote the standard norm on $\mathbb{R}^n$, $n\in\mathbb{N}$. Let $H^{-1}(G)$ be the dual space of $H_0^1(G)$, and the normal of $H^{-1}(G)$ be denoted as $\|\cdot\|_{H^{-1}}$. Denote $\langle\cdot,\cdot\rangle$ the inner product of $L^2(G)$, $\langle\cdot,\cdot\rangle_H$ the inner product of $H_0^1(G)$, and $\langle\cdot,\cdot\rangle_{H^{-1},H}$ the duality pairing between $H^{-1}(G)$ and $H_0^1(G)$. For any $\boldsymbol{u}\in L^2(G)$, there exists an $\boldsymbol{u}'\in H^{-1}(G)$, such that $\langle \boldsymbol{u}',\boldsymbol{v}\rangle_{H^{-1},H} = \langle \boldsymbol{u},\boldsymbol{v}\rangle$ for all $\boldsymbol{v}\in H_0^1(G)$. The mapping $\boldsymbol{u}\to\boldsymbol{u}'$ is linear, injective, compact and continuous, and we can identify $\boldsymbol{u}'$ with $\boldsymbol{u}$. In this sense, we identify $\left(L^2(G)\right)^{-1}$ with $L^2(G)$. Hence $L^2(G)$ is a dense subset of $H^{-1}(G)$ and we have evolution triple

$$H_0^1(G)\subset L^2(G)\subset H^{-1}(G).$$

Hence $\|\boldsymbol{u}\|_{H^{-1}} \le c|\boldsymbol{u}| \le k\|\boldsymbol{u}\|$ for all $\boldsymbol{u}\in H_0^1(G)$ and some constants c and k.

Define the following operator

$$\mathcal{L}\boldsymbol{u} \triangleq -\sum_{i,j=1}^{d} \boldsymbol{u}_{x_i x_j}$$

for any $\boldsymbol{u}\in L^2(G)$. Let $\{\lambda_k\}_{k=1}^{\infty}$ be a family of nondecreasing unbounded positive numbers such that for each k, λ_k is an eigenvalue of the operator $\mathcal{L}$. For every $k\in\mathbb{N}$, let $e_k\in H_0^1(G)$ be a corresponding eigenfunction such that $\{e_k\}_{k=1}^{\infty}$ forms an orthonormal basis of $L^2(G)$. Let $\{q_i\}_{i=1}^{\infty}$ be a family of positive numbers such that $\sum_{i=1}^{\infty} q_i < \infty$. Let our Wiener process W be defined as

$$W(t,x) \triangleq \sum_{i=1}^{\infty} \sqrt{q_i} B^i(t) \boldsymbol{e}_i(x),$$

where $\{B^i(t)\}$ is a sequence of iid Brownian motions in $\mathbb{R}$. Let Q be the operator from $L^2(G)$ to $L^2(G)$ such that $Q\boldsymbol{e}_i = q_i\boldsymbol{e}_i$ for all i. Note that Q is a self-adjoint and positive definite nuclear operator, and W is a $L^2(G)$-valued Q Wiener process. Let L_Q denote the space of all linear operators E such that $EQ^{\frac{1}{2}}$ is a Hilbert-Schmidt operator from $L^2(G)$ to $L^2(G)$, with the inner product

$$\langle F,G\rangle_{L_Q} = tr\left(FQG^*\right) = tr\left(GQF^*\right)$$

for all F and $G \in L_Q$.

Now let us suppress variables in $\boldsymbol{u}$, $\boldsymbol{v}$ and W, and rewrite the system as follows:

$$\begin{cases} \partial_t \boldsymbol{u} = -a\mathcal{L}\boldsymbol{u}\mathrm{d}t + b\left(t,x,\boldsymbol{u},\boldsymbol{v},Z\right)\mathrm{d}t + \sigma\left(t,x,\boldsymbol{u},\boldsymbol{v},Z\right)\mathrm{d}W \\ \partial_t \boldsymbol{v} = h\mathcal{L}\boldsymbol{v}\mathrm{d}t + k\left(t,x,\boldsymbol{u},\boldsymbol{v},Z\right)\mathrm{d}t + Z\mathrm{d}W \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x) \text{ and } \boldsymbol{v}(T,x) = g\left(\boldsymbol{u}(T,x)\right), t\in[0,T], x\in G. \end{cases} \tag{1.2}$$

The rest of the paper is organized as follows. The assumptions on monotonicity conditions and Lipschitz continuity are introduced in Section 2, and the well-posedness of system (1.2) are obtained. In Section 3, in the presence of coercivity assumption, similar results can be shown for system (3.1). In this system, second-order terms in (1.2) have been replaced by continuous linear operators. In sections 4 and 5, a more general system of FBSPDEs (4.1) has been introduced, and the existence and uniqueness of solutions of the system have been obtained.

2. Assumptions

Let us first provide the definition of an adaptive solution to (1.2).

Definition 2.1 *A triple* $(\boldsymbol{u},\boldsymbol{v},Z)$ *is said to be a solution of* (1.2) *if it satisfies* (1.2) *P-a.s. and it is in the space*

$$L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,T;H^1_0(G)\right)\right)\times L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,T;H^1_0(G)\right)\right)\times L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,T;L_Q\right)\right).$$

Let us assume the following assumptions.

(A.1) For fixed $u,v\in H^1_0(G)$ and $z\in L_Q$, the triple (b,k,σ) is in

$$L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,T;H^{-1}(G)\right)\right)\times L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,T;H^{-1}(G)\right)\right)\times L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,T;L_Q\right)\right),$$

and $g(u)$ is in $L^2_{\mathcal{F}_T}\left(\Omega;L^2(G)\right)$.

(A.2) There exists a constant $c_1>0$, such that for every $t>0$, $u,u',v,v'\in H^1_0(G)$, and $z,z'\in L_Q$,

$$\begin{aligned} &\left\langle b\left(t,\cdot,u,v,z\right)-b\left(t,\cdot,u',v,z\right),u-u'\right\rangle_{H^{-1},H} \\ &+\left\|\sigma\left(t,\cdot,u,v,z\right)-\sigma\left(t,\cdot,u',v,z\right)\right\|^2_{L_Q} \\ &\le -c_1\left\|u-u'\right\|^2, \end{aligned}$$

and

$$-\left\langle k\left(t,\cdot,u,v,z\right)-k\left(t,\cdot,u,v',z'\right),v-v'\right\rangle_{H^{-1},H} \le -c_1\left(\left\|v-v'\right\|^2+\left\|z-z'\right\|^2_{L_Q}\right).$$

(A.3) There exist constants $c_2, c_1 > 0$, such that for every $t > 0$, $u, u', v, v' \in H_0^1(G)$ and $z, z' \in L_Q$,

$$\|b(t,\cdot,u,v,z) - b(t,\cdot,u',v',z')\|_{H^{-1}} + \|\sigma(t,\cdot,u,v,z) - \sigma(t,\cdot,u',v',z')\|_{L_Q}$$
$$+|k(t,\cdot,u,v,z) - k(t,\cdot,u',v',z')|_{H^{-1}}$$
$$\le c_2\left(\|u-u'\| + \|v-v'\| + \|z-z'\|_{L_Q}\right)$$

and

$$|g(u) - g(u')| \le c_3 |u-u'|, \text{ P-a.s.}$$

Lemma 2.2. *Assume assumptions* $(A.1)$ *and* $(A.3)$. *Let* (u,v,z) *be a triple in*

$$L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,T; H_0^1(G)\right)\right) \times L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,T; H_0^1(G)\right)\right) \times L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,T; L_Q\right)\right).$$

(1) If

$$c_2^2 < \frac{a^2}{1+a},$$

the forward equation

$$\begin{cases} \partial_t \boldsymbol{u} = -a\mathcal{L}\boldsymbol{u}\mathrm{d}t + b(t,x,\boldsymbol{u},v,z)\mathrm{d}t + \sigma(t,x,\boldsymbol{u},v,z)\mathrm{d}W \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x), t \in [0,T], x \in G \end{cases}$$

has a unique adapted solution $\boldsymbol{u}$ in

$$L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,T; H_0^1(G)\right)\right).$$

(2) If

$$c_2^2 < \frac{\min(h,h^2)}{2},$$

the backward equation

$$\begin{cases} \partial_t \boldsymbol{v} = h\mathcal{L}\boldsymbol{v}\mathrm{d}t + k(t,x,u,\boldsymbol{v},Z)\mathrm{d}t + Z\mathrm{d}W \\ \boldsymbol{v}(T,x) = g(u(T,x)), t \in [0,T], x \in G \end{cases}$$

has a unique adapted solution $(\boldsymbol{v}, Z)$ in

$$L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,T; H_0^1(G)\right)\right) \times L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,T; L_Q\right)\right).$$

Proof. We will prove the lemma by an application of the contraction mapping theorem. Let us first prove the existence and uniqueness of a solution of the forward equation. For any $u, u' \in L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,T; H_0^1(G)\right)\right)$, let $\boldsymbol{u}$ and $\boldsymbol{u}'$ be the unique solutions of

$$\begin{cases} \partial_t \boldsymbol{u} = -a\mathcal{L}\boldsymbol{u}\mathrm{d}t + b(t,x,u,v,z)\mathrm{d}t + \sigma(t,x,u,v,z)\mathrm{d}W \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x), t \in [0,T], x \in G, \end{cases}$$

and

$$\begin{cases} \partial_t \boldsymbol{u}' = -a\mathcal{L}\boldsymbol{u}'\mathrm{d}t + b(t,x,u',v,z)\mathrm{d}t + \sigma(t,x,u',v,z)\mathrm{d}W \\ \boldsymbol{u}'(0,x) = \boldsymbol{u}_0(x), t \in [0,T], x \in G, \end{cases}$$

respectively. Define

$$\hat{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}', \ \hat{u} = u - u',$$

$$\hat{b}(t,x) = b(t,x,u,v,z) - b(t,x,u',v,z),$$

$$\hat{\sigma}(t,x) = \sigma(t,x,u,v,z) - \sigma(t,x,u',v,z).$$

Applying the Itô formula to $|\hat{\boldsymbol{u}}|^2$ and assumption (A.3) to get

$$\begin{aligned} E\left|\hat{\boldsymbol{u}}(T,x)\right|^2 &= E\int_0^T 2\left\langle -a\mathcal{L}\hat{\boldsymbol{u}} + \hat{b}, \hat{\boldsymbol{u}}\right\rangle \mathrm{d}t + E\int_0^T \|\hat{\sigma}\|_{L_Q}^2 \,\mathrm{d}t \\ &\le -2aE\int_0^T \|\hat{\boldsymbol{u}}\|^2 \,\mathrm{d}t + 2c_2 E\int_0^T \|\hat{u}\|\|\hat{\boldsymbol{u}}\| \,\mathrm{d}t + c_2^2 E\int_0^T \|\hat{u}\|^2 \,\mathrm{d}t \\ &\le -aE\int_0^T \|\hat{\boldsymbol{u}}\|^2 \,\mathrm{d}t + c_2^2\left(\frac{1}{a}+1\right)E\int_0^T \|\hat{u}\|^2 \,\mathrm{d}t. \end{aligned}$$

Thus if

$$c_2^2 < \frac{a^2}{1+a},$$

the mapping $u \mapsto \boldsymbol{u}$ is a contraction, and the existence and uniqueness of a solution is guaranteed.

Now let us study the backward system. For any (v,z) and (v',z') in

$$L_{\mathcal{F}}^2\left(\Omega; L^2\left(0,T;H_0^1(G)\right)\right) \times L_{\mathcal{F}}^2\left(\Omega; L^2\left(0,T;L_Q\right)\right),$$

let $(\boldsymbol{v},Z)$ and $(\boldsymbol{v}',Z')$ be solutions of

$$\begin{cases} \partial_t \boldsymbol{v} = h\mathcal{L}\boldsymbol{v}\mathrm{d}t + k(t,x,u,v,z)\mathrm{d}t + Z\mathrm{d}W \\ \boldsymbol{v}(T,x) = g(u(T,x)), t \in [0,T], x \in G, \end{cases}$$

and

$$\begin{cases} \partial_t \boldsymbol{v}' = h\mathcal{L}\boldsymbol{v}'\mathrm{d}t + k(t,x,u,v',z')\mathrm{d}t + Z'\mathrm{d}W \\ \boldsymbol{v}'(T,x) = g(u(T,x)), t \in [0,T], x \in G, \end{cases}$$

respectively. Define

$$\hat{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{v}', \ \hat{v} = v - v', \ \hat{Z} = Z - Z', \ \hat{z} = z - z',$$

$$\hat{k}(t,x) = k(t,x,u,v,z) - k(t,x,u,v',z').$$

An application of the Itô formula to $|\hat{\boldsymbol{v}}|^2$ and assumption (A.3) yield

$$\begin{aligned} E\left|\hat{\boldsymbol{v}}(0,x)\right|^2 + E\int_0^T \left\|\hat{Z}\right\|_{L_Q}^2 \mathrm{d}t &= E\int_0^T 2\left\langle -h\mathcal{L}\hat{\boldsymbol{v}} + \hat{k}, \hat{\boldsymbol{v}}\right\rangle \mathrm{d}t \\ &\le -2hE\int_0^T \|\hat{\boldsymbol{v}}\|^2 \,\mathrm{d}t + 2c_2 E\int_0^T \left(\|\hat{v}\| + \|\hat{z}\|\right)\|\hat{\boldsymbol{v}}\| \mathrm{d}t \\ &\le -hE\int_0^T \|\hat{\boldsymbol{v}}\|^2 \,\mathrm{d}t + \frac{2c_2^2}{h} E\int_0^T \left(\|\hat{v}\|^2 + \|\hat{z}\|^2\right)\mathrm{d}t. \end{aligned}$$

Clearly that if

$$c_2^2 < \frac{\min\left(h,h^2\right)}{2},$$

the backward admits a unique adapted solution. □

Remark 2.3. Lemma 2.2 is a decoupling device: with (v,z) frozen, the forward equation for $\boldsymbol{u}$ and the backward equation for $(\boldsymbol{v},Z)$ are, individually, classical monotone SPDEs of the type first treated by Pardoux [3] and Krylov-Rozovskii [4], and assumptions (A.1) and (A.3) place b, σ, k exactly in the framework for which the well-posedness of such single-direction (forward-only or backward-only) SPDEs is classical. The contraction argument above is the standard way of recovering that well-posedness once the coefficients are frozen in the "other" variable; it plays the same role here that Picard-iteration on the frozen coefficients plays for the linear system in Lemma 4.2 below. The size restrictions on c_2^2 simply say that the Lipschitz constant of b (respectively k) in u (respectively (v,z)) must not overwhelm the coercivity constant a (respectively h) of the leading operator $\mathcal{L}$ - this is the usual competition, in the monotonicity method, between the dissipation supplied by the principal part and the growth of the lower-order coupling terms.

Under certain regularity conditions, we are able to prove the following result.

Theorem 2.4. *Suppose assumptions* $(A.1)$, $(A.2)$ *and* $(A.3)$ *hold. In addition, suppose that*

$$c_2^2 < \frac{a^2}{1+a}, \quad c_2^2 < \frac{\min\left(h,h^2\right)}{2} \quad \textit{and} \quad \frac{c_2^4\left(1+2c_1\right)}{8c_1^2} < ah \le \frac{c_2^2}{4c_3^2}. \tag{2.1}$$

The system (1.2) admits a unique adapted solution in the space

$$L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,T;H_0^1(G)\right)\right)\times L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,T;H_0^1(G)\right)\right)\times L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,T;L_Q\right)\right).$$

Proof. We are going to prove the theorem using the contraction mapping theorem. Let (v,z) and (v',z') be any two pairs from

$$L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,T;H_0^1(G)\right)\right)\times L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,T;L_Q\right)\right).$$

Suppose that $\boldsymbol{u}$ and $\boldsymbol{u}'$ be the solutions of the forward equations

$$\begin{cases}\partial_t \boldsymbol{u} = -a\mathcal{L}\boldsymbol{u}\mathrm{d}t + b\left(t,x,\boldsymbol{u},v,z\right)\mathrm{d}t + \sigma\left(t,x,\boldsymbol{u},v,z\right)\mathrm{d}W\\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x),\, t\in[0,T],\, x\in G\end{cases}$$

and

$$\begin{cases}\partial_t \boldsymbol{u}' = -a\mathcal{L}\boldsymbol{u}'\mathrm{d}t + b\left(t,x,\boldsymbol{u}',v',z'\right)\mathrm{d}t + \sigma\left(t,x,\boldsymbol{u}',v',z'\right)\mathrm{d}W\\ \boldsymbol{u}'(0,x) = \boldsymbol{u}_0(x),\, t\in[0,T],\, x\in G,\end{cases}$$

respectively. Then by solving backward equations

$$\begin{cases}\partial_t \boldsymbol{v} = h\mathcal{L}\boldsymbol{v}\mathrm{d}t + k\left(t,x,\boldsymbol{u},\boldsymbol{v},Z\right)\mathrm{d}t + Z\mathrm{d}W\\ \boldsymbol{v}(T,x) = g\left(\boldsymbol{u}(T,x)\right),\, t\in[0,T],\, x\in G\end{cases}$$

and

$$\begin{cases}\partial_t \boldsymbol{v}' = h\mathcal{L}\boldsymbol{v}'\mathrm{d}t + k\left(t,x,\boldsymbol{u}',\boldsymbol{v}',Z'\right)\mathrm{d}t + Z'\mathrm{d}W\\ \boldsymbol{v}'(T,x) = g\left(\boldsymbol{u}'(T,x)\right),\, t\in[0,T],\, x\in G,\end{cases}$$

one obtains solutions $(\boldsymbol{v},Z)$ and $(\boldsymbol{v}',Z')$. Let

$$\hat{u} = u - u',\ \hat{v} = v - v',\ \hat{z} = z - z',$$

$$\hat{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}',\ \hat{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{v}' \text{ and } \hat{Z} = Z - Z'.$$

The remaining of the theorem is to show that

$$E\int_0^T \left\{ \|\hat{\boldsymbol{v}}\|^2 + \|\hat{Z}\|_{L_Q}^2 \right\} \mathrm{d}s \le qE\int_0^T \left\{ \|\hat{v}\|^2 + \|\hat{z}\|_{L_Q}^2 \right\} \mathrm{d}s$$

for some $q < 1$. An application of the Itô formula yields

$$\begin{aligned}
E\left|\hat{\boldsymbol{u}}(T,x)\right|^2 &= E\int_0^T 2\left\langle -a\mathcal{L}\hat{\boldsymbol{u}} + \left[b(t,x,\boldsymbol{u},v,z) - b(t,x,\boldsymbol{u}',v',z')\right], \hat{\boldsymbol{u}} \right\rangle_{H^{-1},H} \mathrm{d}t \\
&\quad + E\int_0^T \left\|\sigma(t,x,\boldsymbol{u},v,z) - \sigma(t,x,\boldsymbol{u}',v',z')\right\|_{L_Q}^2 \mathrm{d}t \\
&\le -2aE\int_0^T \|\hat{\boldsymbol{u}}\|^2 \mathrm{d}t + 2E\int_0^T \left\{ \left\langle b(t,x,\boldsymbol{u},v,z) - b(t,x,\boldsymbol{u}',v,z), \hat{\boldsymbol{u}} \right\rangle_{H^{-1},H} \right. \\
&\quad \left. + \left\|\sigma(t,x,\boldsymbol{u},v,z) - \sigma(t,x,\boldsymbol{u}',v,z)\right\|_{L_Q}^2 \right\} \mathrm{d}t \\
&\quad + 2E\int_0^T \left\{ \left\|b(t,x,\boldsymbol{u}',v,z) - b(t,x,\boldsymbol{u}',v',z')\right\|_{H^{-1}} \|\hat{\boldsymbol{u}}\| \right. \\
&\quad \left. + \left\|\sigma(t,x,\boldsymbol{u}',v,z) - \sigma(t,x,\boldsymbol{u}',v',z')\right\|_{L_Q}^2 \right\} \mathrm{d}t.
\end{aligned}$$

Thus assumptions (A.2) and (A.3) imply that

$$\begin{aligned}
&E\left|\hat{\boldsymbol{u}}(T,x)\right|^2 + 2aE\int_0^T \|\hat{\boldsymbol{u}}\|^2 \mathrm{d}t \\
&\le \frac{2c_2^2}{c_1}(1+2c_1) E\int_0^T \left\{ \|\hat{v}\|^2 + \|\hat{z}\|_{L_Q}^2 \right\} \mathrm{d}t.
\end{aligned} \tag{2.2}$$

Then utilizing the Itô formula, assumptions (A.2) and (A.3), one gets

$$\begin{aligned}
&E\left|\hat{\boldsymbol{v}}(0,x)\right|^2 + E\int_0^T \|\hat{Z}\|_{L_Q}^2 \mathrm{d}t \\
&= E\left|\hat{\boldsymbol{v}}(T,x)\right|^2 - E\int_0^T 2\left\langle h\mathcal{L}\hat{\boldsymbol{v}} + \left[k(t,x,\boldsymbol{u},\boldsymbol{v},Z) - k(t,x,\boldsymbol{u}',\boldsymbol{v}',Z')\right], \hat{\boldsymbol{v}} \right\rangle_{H^{-1},H} \mathrm{d}t \\
&\le E\left|\hat{\boldsymbol{v}}(T,x)\right|^2 - 2hE\int_0^T \|\hat{\boldsymbol{v}}\|^2 \mathrm{d}t \\
&\quad + E\int_0^T -2\left\langle k(t,x,\boldsymbol{u},\boldsymbol{v},Z) - k(t,x,\boldsymbol{u},\boldsymbol{v}',Z'), \hat{\boldsymbol{v}} \right\rangle_{H^{-1},H} \mathrm{d}t \\
&\quad - 2E\int_0^T \left\|k(t,x,\boldsymbol{u},\boldsymbol{v}',Z') - k(t,x,\boldsymbol{u}',\boldsymbol{v}',Z')\right\|_{H^{-1}} \|\hat{\boldsymbol{v}}\| \mathrm{d}t \\
&\le -2c_1 E\int_0^T \left\{ \|\hat{\boldsymbol{v}}\|^2 + \|\hat{Z}\|_{L_Q}^2 \right\} \mathrm{d}t + \frac{c_2^2}{2h} E\int_0^T \|\hat{\boldsymbol{u}}\|^2 \mathrm{d}t.
\end{aligned} \tag{2.3}$$

Substituting (2.2) into (2.3), and applying (A.3) to obtain

$$\begin{aligned}
E\int_0^T \left\{ \|\hat{\boldsymbol{v}}\|^2 + \|\hat{Z}\|_{L_Q}^2 \right\} \mathrm{d}t &\le \left(\frac{1}{2c_1}\right)\left(c_3^2 - \frac{c_2^2}{4ah}\right) E\left|\hat{\boldsymbol{u}}(T,x)\right|^2 \\
&\quad + \frac{c_2^4(1+2c_1)}{8ahc_1^2} E\int_0^T \left\{ \|\hat{v}\|^2 + \|\hat{z}\|_{L_Q}^2 \right\} \mathrm{d}t.
\end{aligned}$$

The two parts of condition (2.1) now play complementary roles: the requirement $ah \le c_2^2/(4c_3^2)$ forces the coefficient $c_3^2 - c_2^2/(4ah)$ of $E\left|\hat{\boldsymbol{u}}(T,x)\right|^2$ to be non-positive, so that term may be dropped, while the requirement $c_2^4(1+2c_1)/(8c_1^2) < ah$ makes the remaining coefficient $c_2^4(1+2c_1)/(8ahc_1^2)$

strictly less than 1. Hence

$$E\int_0^T \left\{ \|\hat{\boldsymbol{v}}\|^2 + \|\hat{Z}\|_{L_Q}^2 \right\} \mathrm{d}t \le qE\int_0^T \left\{ \|\hat{v}\|^2 + \|\hat{z}\|_{L_Q}^2 \right\} \mathrm{d}t$$

for some $q \in (0,1)$, and the mapping $(v,z) \mapsto (\boldsymbol{v}, Z)$ is a contraction on $L_{\mathcal{F}}^2\left(\Omega; L^2\left(0,T; H_0^1(G)\right)\right) \times L_{\mathcal{F}}^2\left(\Omega; L^2\left(0,T; L_Q\right)\right)$. Thus by (2.1) and the contraction mapping theorem, system (1.2) admits a unique adapted solution in the desired space. □

Remark 2.5. Condition (2.1) should be read as a window for the product ah of the two diffusion constants, sandwiched between a lower bound coming from the forward-to-backward coupling (through b, σ) and an upper bound coming from the terminal coupling g: ah must be large enough that the iteration $(v,z) \mapsto (\boldsymbol{v}, Z)$ in the proof of Lemma 2.2 contracts, yet not so large that the terminal condition $\boldsymbol{v}(T,x) = g(\boldsymbol{u}(T,x))$ destroys that contraction by feeding too much of the forward solution's fluctuation back into $\boldsymbol{v}$. This mirrors the well-known tension in fully coupled FBSDEs between the length of the time horizon and the size of the coupling: here the roles of "short horizon" and "weak coupling" are played by the sizes of a, h relative to c_2 and c_3.

3. Coercivity Assumption

Let A and B be two continuous linear operators from $H_0^1(G)$ to $H^{-1}(G)$ with domains dense in $L^2(G)$, such that the following coercivity assumption is held.

(A.4) For every $u \in H_0^1(G)$, there exist constants $\alpha > 0$, $\beta > 0$ and γ, such that

$$\langle A(u), u \rangle_{H^{-1},H} \le -\alpha \|u\|^2 + \gamma |u|^2,$$

$$\langle -B(u), u \rangle_{H^{-1},H} \le -\beta \|u\|^2 - \gamma |u|^2.$$

Let us modify system (1.2) as follows to make it more general:

$$\begin{cases} \partial_t \boldsymbol{u} = A(\boldsymbol{u})\mathrm{d}t + b(t,x,\boldsymbol{u},\boldsymbol{v},Z)\mathrm{d}t + \sigma(t,x,\boldsymbol{u},\boldsymbol{v},Z)\mathrm{d}W \\ \partial_t \boldsymbol{v} = B(\boldsymbol{v})\mathrm{d}t + k(t,x,\boldsymbol{u},\boldsymbol{v},Z)\mathrm{d}t + Z\mathrm{d}W \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x) \text{ and } \boldsymbol{v}(T,x) = g(\boldsymbol{u}(T,x)),\ t \in [0,T],\ x \in G. \end{cases} \tag{3.1}$$

The following result is quite straightforward, and it concludes this section.

Theorem 3.1. *Suppose assumptions* (*A*.1), (*A*.2), (*A*.3) *and* (*A*.4) *hold. In addition, suppose that*

$$c_2^2 < \frac{\alpha^2}{1+\alpha},\quad c_2^2 < \frac{\min(\beta,\beta^2)}{2} \quad \textit{and} \quad \frac{c_2^4(1+2c_1)}{8c_1^2} < \alpha\beta \le \frac{c_2^2}{4c_3^2}. \tag{3.2}$$

The system (3.1) admits a unique adapted solution in the space

$$L_{\mathcal{F}}^2\left(\Omega; L^2\left(0,T; H_0^1(G)\right)\right) \times L_{\mathcal{F}}^2\left(\Omega; L^2\left(0,T; H_0^1(G)\right)\right) \times L_{\mathcal{F}}^2\left(\Omega; L^2\left(0,T; L_Q\right)\right).$$

Proof. Simply we apply the Itô formula to $\mathrm{e}^{-\gamma t}|\hat{\boldsymbol{u}}|^2$ and $\mathrm{e}^{-\gamma t}|\hat{\boldsymbol{v}}|^2$. The rest of the proof is very similar to the proof of Theorem 2.4. □

Remark 3.2. The exponential weights $\mathrm{e}^{-\gamma t}$ absorb the lower-order (non-coercive) part $\gamma|u|^2$ that (A.4) allows A and B to carry; this is the usual device for passing from a strictly coercive operator such as $-a\mathcal{L}$ in (1.2) to a merely coercive-up-to-a-shift operator A. Theorem 3.1 therefore covers, in addition to second-order elliptic operators, e.g., non-symmetric or lower-order perturbations of $\mathcal{L}$, at the price of only bookkeeping the extra constant γ.

4. Generalization of the System

In the next two sections we are going to consider a more general system

$$\begin{cases} \partial_t \boldsymbol{u} = S(t,x,X)\mathrm{d}t + \sigma(t,x,X)\mathrm{d}W \\ \partial_t \boldsymbol{v} = T(t,x,X)\mathrm{d}t + Z\mathrm{d}W \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x) \text{ and } \boldsymbol{v}(T,x) = g(\boldsymbol{u}(T,x)), \end{cases} \tag{4.1}$$

with the following assumptions.

(B.1) For fixed $u,v \in H_0^1(G)$ and $z \in L_Q$, the triple (S,T,σ) is in $L^2_{\mathcal{F}}(\Omega; L^2(0,T;H^{-1}(G))) \times L^2_{\mathcal{F}}(\Omega; L^2(0,T;H^{-1}(G))) \times L^2_{\mathcal{F}}(\Omega; L^2(0,T;L_Q))$,

and $g(u)$ is in $L^2_{\mathcal{F}_T}(\Omega; L^2(G))$.

(B.2) There exists a constant $c_1 > 0$, such that for every $t > 0$, $u,u',v,v' \in H_0^1(G)$ and $z,z' \in L_Q$,

$$\begin{aligned} &\|S(t,\cdot,u,v,z) - S(t,\cdot,u',v',z')\|_{H^{-1}} + \|\sigma(t,\cdot,u,v,z) - \sigma(t,\cdot,u',v',z')\|_{L_Q} \\ &+ \|T(t,\cdot,u,v,z) - T(t,\cdot,u',v',z')\|_{H^{-1}} \\ &\le c_1\left(\|u-u'\| + \|v-v'\| + \|z-z'\|_{L_Q}\right) \end{aligned}$$

and

$$|g(u) - g(u')| \le c_1|u-u'|, \ \text{P-a.s.}$$

(B.3) There exists a constant $c_2 > 0$, such that for every $t > 0$, $u,u',v,v' \in H_0^1(G)$, and $z,z' \in L_Q$,

$$\begin{aligned} &\langle S(t,\cdot,u,v,z) - S(t,\cdot,u',v',z'), v-v'\rangle_{H^{-1},H} \\ &+ \langle \sigma(t,\cdot,u,v,z) - \sigma(t,\cdot,u',v',z'), z-z'\rangle_{L_Q} \\ &+ \langle T(t,\cdot,u,v,z) - T(t,\cdot,u',v',z'), u-u'\rangle_{H^{-1},H} \\ &\le -c_2\left(\|u-u'\|^2 + \|v-v'\|^2 + \|z-z'\|^2_{L_Q}\right) \end{aligned}$$

and

$$\langle g(u) - g(u'), u-u'\rangle \ge c_2|u-u'|^2, \ \text{P-a.s.}$$

Let us prove some preliminary results.

Lemma 4.1. *If a nonnegative sequence* $\{a_i\}_{i=0}^{\infty}$ *satisfies*

$$a_{i+1} \le \frac{1}{4}a_i + \frac{1}{8}a_{i-1}$$

for all $i \ge 1$, *then there exists a constant* c, *such that*

$$a_i \le \frac{c}{2^i}$$

for all $i \ge 0$.

Lemma 4.2. *For any triple* (S_0, T_0, σ_0) *in the space*

$$L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,T; H^{-1}(G)\right)\right) \times L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,T; H^{-1}(G)\right)\right) \times L^2_{\mathcal{F}}\left(\Omega; L^2\left(0,T; L_Q\right)\right),$$

and $g_0 \in L^2_{\mathcal{F}_T}\left(\Omega; L^2(G)\right)$, *the following linear system has a unique adapted solution* $(\boldsymbol{u}, \boldsymbol{v}, Z)$.

$$\begin{cases} \partial_t \boldsymbol{u}(t,x) = \left(-\mathcal{L}\boldsymbol{v}(t,x) + S_0(t,x)\right)\mathrm{d}t + \left(-Z(t,x) + \sigma_0(t,x)\right)\mathrm{d}W(t,x) \\ \partial_t \boldsymbol{v}(t,x) = \left(-\mathcal{L}\boldsymbol{u}(t,x) + T_0(t,x)\right)\mathrm{d}t + Z(t,x)\mathrm{d}W(t,x) \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x) \text{ and } \boldsymbol{v}(T,x) = \boldsymbol{u}(T,x) + g_0(x), \end{cases} \tag{4.2}$$

for $t \in [0,T]$ and $x \in G$.

Proof. We will show the existence of this linear forward backward stochastic partial differential equations in two steps. First for the existence and uniqueness of the solution of

$$\begin{cases} \partial_t \boldsymbol{v}'(t,x) = \left(\mathcal{L}\boldsymbol{v}'(t,x) + T_0(t,x) - S_0(t,x)\right)\mathrm{d}t \\ \qquad\qquad\qquad + \left(2Z(t,x) - \sigma_0(t,x)\right)\mathrm{d}W(t,x) \\ \boldsymbol{v}'(T,x) = g_0(x),\ t \in [0,T],\ x \in G \end{cases}$$

is guaranteed: this is a linear backward SPDE with a coercive principal part $\mathcal{L}$ and coefficients that are, trivially, Lipschitz and monotone, so its well-posedness follows from the classical monotone-operator theory for backward SPDEs of Pardoux [3] and Krylov-Rozovskii [4] (equivalently, this is the special case $b \equiv 0$, $\sigma \equiv 0$ of Lemma 2.2(2) with u suppressed). Also it is easy to see that

$$\begin{cases} \partial_t \boldsymbol{u}(t,x) = \left(-\mathcal{L}\boldsymbol{u}(t,x) - \mathcal{L}\boldsymbol{v}'(t,x) + S_0(t,x)\right)\mathrm{d}t + \left(-Z(t,x) + \sigma_0(t,x)\right)\mathrm{d}W(t,x) \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x),\ t \in [0,T],\ x \in G \end{cases}$$

yields a unique adapted solution $\boldsymbol{u}$, by the same classical theory applied to the (now linear, coercive) forward equation, with $\boldsymbol{v}'$ playing the role of a fixed inhomogeneous term. Let $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{v}'$. Clearly $(\boldsymbol{u}, \boldsymbol{v}, Z)$ is a solution of (4.2).

Now let us show the uniqueness. For simplicity, we suppress the variables t and x. Suppose $(\boldsymbol{u}, \boldsymbol{v}, Z)$ and $(\boldsymbol{u}', \boldsymbol{v}', Z')$ are two solutions. Let $\hat{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}'$, $\hat{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{v}'$, and $\hat{Z} = Z - Z'$. Applying the Itô formula to $\langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}} \rangle$ to get

$$\partial_t \langle \hat{\boldsymbol{u}}, \hat{\boldsymbol{v}} \rangle = -\left(\|\hat{\boldsymbol{u}}\|^2 + \|\hat{\boldsymbol{v}}\|^2 + \|\hat{Z}\|^2_{L_Q}\right)\mathrm{d}t + \left\langle -\hat{Z}^*(\hat{\boldsymbol{v}}) + \hat{Z}^*(\hat{\boldsymbol{u}}), \mathrm{d}W \right\rangle.$$

Here we used the fact that $\langle \mathcal{L}\boldsymbol{u}, \boldsymbol{u} \rangle_{H^{-1},H} = \|\boldsymbol{u}\|^2$ for all $\boldsymbol{u} \in H^1_0(G)$. Thus

$$E\left|\hat{\boldsymbol{u}}(T,x)\right|^2 = -E\int_0^T \left(\|\hat{\boldsymbol{u}}\|^2 + \|\hat{\boldsymbol{v}}\|^2 + \|\hat{Z}\|^2_{L_Q}\right)\mathrm{d}t,$$

and

$$(\boldsymbol{u}, \boldsymbol{v}, Z) = (\boldsymbol{u}', \boldsymbol{v}', Z'), \text{ P-a.s., for every } t \ge 0 \text{ and } x \in G.$$

***Remark* 4.3.** Lemma 4.2 isolates the one fully coupled, but linear, FBSPDE that anchors the method of continuation used below: system (4.2) is exactly the $\alpha = 0$ member of the family (4.3) introduced next. The strategy of Sections 4 and 5 is to move α from 0 to 1 in small, uniform steps δ, at each step reducing solvability of the (nonlinear) system at parameter $\alpha+\delta$ to solvability at parameter α via a fixed-point argument; the uniform size of δ, guaranteed by Lemma 4.4 below, is what allows finitely many such steps to reach $\alpha = 1$.

For simplicity, let us denote $(\boldsymbol{u}, \boldsymbol{v}, Z)$ by X, and suppress variables t and x. Let us define another forward backward stochastic partial differential equations as follows:

$$\begin{cases} \partial_t \boldsymbol{u} = \left(S^{\alpha}(t,x,X) + S_0\right)\mathrm{d}t + \left(\sigma^{\alpha}(t,x,X) + \sigma_0\right)\mathrm{d}W \\ \partial_t \boldsymbol{v} = \left(T^{\alpha}(t,x,X) + T_0\right)\mathrm{d}t + Z\mathrm{d}W \\ \boldsymbol{u}(0,x) = \boldsymbol{u}_0(x) \text{ and } \boldsymbol{v}(T,x) = g^{\alpha}\left(\boldsymbol{u}(T,x)\right) + g_0(x), \end{cases} \tag{4.3}$$

where

$$S^{\alpha}(t,x,u,v,z) = \alpha S(t,x,u,v,z) - (1-\alpha)\mathcal{L}v,$$

$$T^{\alpha}(t,x,u,v,z) = \alpha T(t,x,u,v,z) - (1-\alpha)\mathcal{L}u,$$

$$\sigma^{\alpha}(t,x,u,v,z) = \alpha \sigma(t,x,u,v,z) - (1-\alpha)z,$$

$$g^{\alpha}(u) = \alpha g(u) + (1-\alpha)u.$$

Lemma 4.4. *Assume assumptions (B.1), (B.2) and (B.3). Suppose that for some* $\alpha \in [0,1)$, *and for any* $(S_0, T_0, \sigma_0, g_0)$ *as in Lemma 4.2, system (4.3) has an adapted solution. Then there exists* $0 < \delta < 1$, *such that for any* $\alpha' \in [\alpha, \alpha+\delta]$. *and for any* $(S_0, T_0, \sigma_0, g_0)$ *as in Lemma 4.2, system (4.3) has an adapted solution.*

Proof. Let δ be a number in $(0,1)$. By the definition of system (4.3), it is easy to see that

$$S^{\alpha+\delta}(t,x,u,v,z) = S^{\alpha}(t,x,u,v,z) + \delta\left(S(t,x,u,v,z) + \mathcal{L}v\right),$$

$$T^{\alpha+\delta}(t,x,u,v,z) = T^{\alpha}(t,x,u,v,z) + \delta\left(T(t,x,u,v,z) + \mathcal{L}u\right),$$

$$\sigma^{\alpha+\delta}(t,x,u,v,z) = \sigma^{\alpha}(t,x,u,v,z) + \delta\left(\sigma(t,x,u,v,z) + z\right),$$

$$g^{\alpha+\delta}(u) = g^{\alpha}(u) + \delta\left(g(u) - u\right).$$

Let X^i denote $(\boldsymbol{u}^i, \boldsymbol{v}^i, Z^i)$ for $i \in \mathbb{N}$ and we set $X^0 = (0,0,0)$. By the assumptions of the lemma, the FBSPDE

$$\begin{cases} \partial_t \boldsymbol{u}^{i+1} = \left(S^{\alpha}(t,x,X^{i+1}) + \delta\left(S(t,x,X^i) + \mathcal{L}\boldsymbol{v}^i\right) + S_0\right)\mathrm{d}t \\ \qquad\qquad + \left(\sigma^{\alpha}(t,x,X^{i+1}) + \delta\left(\sigma(t,x,X^i) + Z^i\right) + \sigma_0\right)\mathrm{d}W \\ \partial_t \boldsymbol{v}^{i+1} = \left(T^{\alpha}(t,x,X^{i+1}) + \delta\left(T(t,x,X^i) + \mathcal{L}\boldsymbol{u}^i\right) + T_0\right)\mathrm{d}t + Z^{i+1}\mathrm{d}W \\ \boldsymbol{u}^{i+1}(0,x) = \boldsymbol{u}_0(x) \\ \boldsymbol{v}^{i+1}(T,x) = g^{\alpha}\left(\boldsymbol{u}^{i+1}(T,x)\right) + \delta\left(g(\boldsymbol{u}^i) - \boldsymbol{u}^i\right) + g_0(x) \end{cases} \tag{4.4}$$

admit an adapted solution $\left(\boldsymbol{u}^{i+1},\boldsymbol{v}^{i+1},Z^{i+1}\right)$ for any nonnegative integer i. We are going to show that $\left\{\left(\boldsymbol{u}^{i},\boldsymbol{v}^{i},Z^{i}\right)\right\}_{i=0}^{\infty}$ is a Cauchy sequence in

$$L_{\mathcal{F}}^{2}\left(\Omega;L^{2}\left(0,T;H_{0}^{1}(G)\right)\right)\times L_{\mathcal{F}}^{2}\left(\Omega;L^{2}\left(0,T;H_{0}^{1}(G)\right)\right)\times L_{\mathcal{F}}^{2}\left(\Omega;L^{2}\left(0,T;L_{Q}\right)\right).$$

Let $\hat{X}^{i}=\left(\hat{\boldsymbol{u}}^{i},\hat{\boldsymbol{v}}^{i},\hat{Z}^{i}\right)=X^{i}-X^{i-1}$ for all $i\in\mathbb{N}$. An application of the Itô formula to $\left\langle\hat{\boldsymbol{v}}^{i+1},\hat{\boldsymbol{u}}^{i+1}\right\rangle$ yields

$$\begin{aligned}
&E\left\langle\hat{\boldsymbol{v}}^{i+1}(T,x),\hat{\boldsymbol{u}}^{i+1}(T,x)\right\rangle\\
&=E\int_{0}^{T}\Big\{\left\langle S^{\alpha}\left(t,x,X^{i+1}\right)-S^{\alpha}\left(t,x,X^{i}\right),\hat{\boldsymbol{v}}^{i+1}\right\rangle_{H^{-1},H}\\
&\quad+\left\langle T^{\alpha}\left(t,x,X^{i+1}\right)-T^{\alpha}\left(t,x,X^{i}\right),\hat{\boldsymbol{u}}^{i+1}\right\rangle_{H^{-1},H}\\
&\quad+\left\langle\sigma^{\alpha}\left(t,x,X^{i+1}\right)-\sigma^{\alpha}\left(t,x,X^{i}\right),\hat{Z}^{i+1}\right\rangle_{L_{Q}}\Big\}\mathrm{d}t\\
&\quad+E\int_{0}^{T}\Big\{\left\langle\delta\left(S\left(t,x,X^{i}\right)-S\left(t,x,X^{i-1}\right)\right),\hat{\boldsymbol{v}}^{i+1}\right\rangle_{H^{-1},H}\\
&\quad+\left\langle\delta\left(T\left(t,x,X^{i}\right)-T\left(t,x,X^{i-1}\right)\right),\hat{\boldsymbol{u}}^{i+1}\right\rangle_{H^{-1},H}\\
&\quad+\left\langle\delta\left(\sigma\left(t,x,X^{i}\right)-\sigma\left(t,x,X^{i-1}\right)\right),\hat{Z}^{i+1}\right\rangle_{L_{Q}}\Big\}\mathrm{d}t\\
&\quad+\delta E\int_{0}^{T}\Big\{\left\langle\mathcal{L}\hat{\boldsymbol{v}}^{i},\hat{\boldsymbol{v}}^{i+1}\right\rangle_{H^{-1},H}+\left\langle\mathcal{L}\hat{\boldsymbol{u}}^{i},\hat{\boldsymbol{u}}^{i+1}\right\rangle_{H^{-1},H}+\left\langle\hat{Z}^{i},\hat{Z}^{i+1}\right\rangle_{L_{Q}}\Big\}\mathrm{d}t.
\end{aligned}$$

By assumption (B.2) and (B.3), one gets

$$\begin{aligned}
&\alpha c_{2}E\left|\hat{\boldsymbol{u}}^{i+1}(T,x)\right|^{2}+(1-\alpha)E\left|\hat{\boldsymbol{u}}^{i+1}(T,x)\right|^{2}\\
&\leq\delta\left(1+c_{1}\right)E\left|\hat{\boldsymbol{u}}^{i}(T,x)\right|\left|\hat{\boldsymbol{u}}^{i+1}(T,x)\right|\\
&\quad-\alpha c_{2}E\int_{0}^{T}\Big\{\left\|\hat{\boldsymbol{u}}^{i+1}\right\|^{2}+\left\|\hat{\boldsymbol{v}}^{i+1}\right\|^{2}+\left\|\hat{Z}^{i+1}\right\|_{L_{Q}}^{2}\Big\}\mathrm{d}t\\
&\quad-(1-\alpha)E\int_{0}^{T}\Big\{\left\|\hat{\boldsymbol{u}}^{i+1}\right\|^{2}+\left\|\hat{\boldsymbol{v}}^{i+1}\right\|^{2}+\left\|\hat{Z}^{i+1}\right\|_{L_{Q}}^{2}\Big\}\mathrm{d}t\\
&\quad+\delta E\int_{0}^{T}\Big\{\left\|S\left(t,x,X^{i}\right)-S\left(t,x,X^{i-1}\right)\right\|_{H^{-1}}\left\|\hat{\boldsymbol{v}}^{i+1}\right\|\\
&\quad+\left\|T\left(t,x,X^{i}\right)-T\left(t,x,X^{i-1}\right)\right\|_{H^{-1}}\left\|\hat{\boldsymbol{v}}^{i+1}\right\|\\
&\quad+\left\|\sigma\left(t,x,X^{i}\right)-\sigma\left(t,x,X^{i-1}\right)\right\|_{L_{Q}}\left\|\hat{Z}^{i+1}\right\|_{L_{Q}}\Big\}\mathrm{d}t\\
&\quad+\delta E\int_{0}^{T}\Big\{\left\|\hat{\boldsymbol{u}}^{i}\right\|\cdot\left\|\hat{\boldsymbol{u}}^{i+1}\right\|+\left\|\hat{\boldsymbol{v}}^{i}\right\|\cdot\left\|\hat{\boldsymbol{v}}^{i+1}\right\|+\left\|\hat{Z}^{i}\right\|_{L_{Q}}\left\|\hat{Z}^{i+1}\right\|_{L_{Q}}\Big\}\mathrm{d}t,
\end{aligned}$$

where

$$\begin{aligned}
&E\int_{0}^{T}\Big\{\left\|S\left(t,x,X^{i}\right)-S\left(t,x,X^{i-1}\right)\right\|_{H^{-1}}\left\|\hat{\boldsymbol{v}}^{i+1}\right\|+\left\|T\left(t,x,X^{i}\right)-T\left(t,x,X^{i-1}\right)\right\|_{H^{-1}}\left\|\hat{\boldsymbol{v}}^{i+1}\right\|\\
&+\left\|\sigma\left(t,x,X^{i}\right)-\sigma\left(t,x,X^{i-1}\right)\right\|_{L_{Q}}\left\|\hat{Z}^{i+1}\right\|_{L_{Q}}\Big\}\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\delta^2}{c_3} E\int_0^T \Big\{ \left\| S\left(t,x,X^i\right) - S\left(t,x,X^{i-1}\right) \right\|^2 + \left\| T\left(t,x,X^i\right) - T\left(t,x,X^{i-1}\right) \right\|^2 \\
&\quad + \left\| \sigma\left(t,x,X^i\right) - \sigma\left(t,x,X^{i-1}\right) \right\|_{L_Q}^2 \Big\} \mathrm{d}t + \frac{c_3}{4} E\int_0^T \left\{ \left\|\hat{\boldsymbol{u}}^{i+1}\right\|^2 + \left\|\hat{\boldsymbol{v}}^{i+1}\right\|^2 + \left\|\hat{Z}^{i+1}\right\|_{L_Q}^2 \right\} \mathrm{d}t \\
&\le \frac{3\delta^2 c_1^2}{c_3} E\int_0^T \left\{ \left\|\hat{\boldsymbol{u}}^{i}\right\|^2 + \left\|\hat{\boldsymbol{v}}^{i}\right\|^2 + \left\|\hat{Z}^{i}\right\|_{L_Q}^2 \right\} \mathrm{d}t \\
&\quad + \frac{c_3}{4} E\int_0^T \left\{ \left\|\hat{\boldsymbol{u}}^{i+1}\right\|^2 + \left\|\hat{\boldsymbol{v}}^{i+1}\right\|^2 + \left\|\hat{Z}^{i+1}\right\|_{L_Q}^2 \right\} \mathrm{d}t.
\end{aligned}
$$

Let $c_3 = \min(1, c_2)$. Using the fact that $\alpha c_2 + 1 - \alpha \ge \min(1, c_2)$, the above inequality becomes

$$
\begin{aligned}
& c_3 E\left|\hat{\boldsymbol{u}}^{i+1}(T,x)\right|^2 + c_3 E\int_0^T \left\{ \left\|\hat{\boldsymbol{u}}^{i+1}\right\|^2 + \left\|\hat{\boldsymbol{v}}^{i+1}\right\|^2 + \left\|\hat{Z}^{i+1}\right\|_{L_Q}^2 \right\} \mathrm{d}t \\
&\le \delta(1+c_1) E\left|\hat{\boldsymbol{u}}^{i}(T,x)\right|\left|\hat{\boldsymbol{u}}^{i+1}(T,x)\right| \\
&\quad + \frac{\delta^2\left(1+3c_1^2\right)}{c_3} E\int_0^T \left\{ \left\|\hat{\boldsymbol{u}}^{i}\right\|^2 + \left\|\hat{\boldsymbol{v}}^{i}\right\|^2 + \left\|\hat{Z}^{i}\right\|_{L_Q}^2 \right\} \mathrm{d}t \\
&\quad + \frac{c_3}{2} E\int_0^T \left\{ \left\|\hat{\boldsymbol{u}}^{i+1}\right\|^2 + \left\|\hat{\boldsymbol{v}}^{i+1}\right\|^2 + \left\|\hat{Z}^{i+1}\right\|_{L_Q}^2 \right\} \mathrm{d}t.
\end{aligned}
$$

Thus

$$
\begin{aligned}
& E\int_0^T \left\{ \left\|\hat{\boldsymbol{u}}^{i+1}\right\|^2 + \left\|\hat{\boldsymbol{v}}^{i+1}\right\|^2 + \left\|\hat{Z}^{i+1}\right\|_{L_Q}^2 \right\} \mathrm{d}t \\
&\le \frac{\delta^2(1+c_1)^2}{2c_3^2} E\left|\hat{\boldsymbol{u}}^{i}(T,x)\right|^2 + \frac{2\delta^2\left(1+3c_1^2\right)}{c_3^2} E\int_0^T \left\{ \left\|\hat{\boldsymbol{u}}^{i}\right\|^2 + \left\|\hat{\boldsymbol{v}}^{i}\right\|^2 + \left\|\hat{Z}^{i}\right\|_{L_Q}^2 \right\} \mathrm{d}t.
\end{aligned}
$$

Now let us find a bound for $E\left|\hat{\boldsymbol{u}}^{i}(T,x)\right|^2$. Applying the Itô formula to $\left|\hat{\boldsymbol{u}}^{i}\right|^2$, utilizing assumption (B.2), and after some calculations, one obtains

$$
\begin{aligned}
E\left|\hat{\boldsymbol{u}}^{i}(T,x)\right|^2 &= E\int_0^T \Big\{ 2\left\langle S^\alpha\left(t,x,X^i\right) - S^\alpha\left(t,x,X^{i-1}\right), \hat{\boldsymbol{u}}^i \right\rangle_{H^{-1},H} \\
&\quad + 2\delta\left\langle S\left(t,x,X^{i-1}\right) - S\left(t,x,X^{i-2}\right) + \mathcal{L}\hat{\boldsymbol{v}}^{i-1}, \hat{\boldsymbol{u}}^i \right\rangle_{H^{-1},H} \\
&\quad + \Big\| \sigma^\alpha\left(t,x,X^i\right) - \sigma^\alpha\left(t,x,X^{i-1}\right) \\
&\quad + \delta\left(\sigma\left(t,x,X^{i-1}\right) - \sigma\left(t,x,X^{i-2}\right) + \hat{Z}^{i-1}\right) \Big\|_{L_Q} \Big\} \mathrm{d}t \\
&\le \delta(1+c_1+6\delta) E\int_0^T \left\{ \left\|\hat{\boldsymbol{u}}^{i-1}\right\|^2 + \left\|\hat{\boldsymbol{v}}^{i-1}\right\|^2 + \left\|\hat{Z}^{i-1}\right\|_{L_Q}^2 \right\} \mathrm{d}t \\
&\quad + (2+5c_1) E\int_0^T \left\{ \left\|\hat{\boldsymbol{u}}^{i}\right\|^2 + \left\|\hat{\boldsymbol{v}}^{i}\right\|^2 + \left\|\hat{Z}^{i}\right\|_{L_Q}^2 \right\} \mathrm{d}t.
\end{aligned}
$$

By carefully choosing a $\delta \in (0,1)$, it is easy to get

$$
\begin{aligned}
& E\int_0^T \left\{ \left\|\hat{\boldsymbol{u}}^{i+1}\right\|^2 + \left\|\hat{\boldsymbol{v}}^{i+1}\right\|^2 + \left\|\hat{Z}^{i+1}\right\|_{L_Q}^2 \right\} \mathrm{d}t \le \frac{1}{4} E\int_0^T \left\{ \left\|\hat{\boldsymbol{u}}^{i-1}\right\|^2 + \left\|\hat{\boldsymbol{v}}^{i-1}\right\|^2 + \left\|\hat{Z}^{i-1}\right\|_{L_Q}^2 \right\} \mathrm{d}t \\
&\quad + \frac{1}{8} E\int_0^T \left\{ \left\|\hat{\boldsymbol{u}}^{i}\right\|^2 + \left\|\hat{\boldsymbol{v}}^{i}\right\|^2 + \left\|\hat{Z}^{i}\right\|_{L_Q}^2 \right\} \mathrm{d}t
\end{aligned}
$$

for all $0<\delta'\le\delta$. Lemma 4.1 shows that $\left\{\left(\boldsymbol{u}^i,\boldsymbol{v}^i,Z^i\right)\right\}_{i=0}^{\infty}$ is a Cauchy sequence in the desired spaces. Let $(\boldsymbol{u},\boldsymbol{v},Z)$ be the limit. Since all the coefficients in (4.3) are continuous under assumption (B.2), we can take the limit on both sides of (4.4), and it is clear that $(\boldsymbol{u},\boldsymbol{v},Z)$ solves (4.3) for $\alpha'=\alpha+\delta'$, which completes the proof. □

5. Existence and Uniqueness

Now we are ready to provide the main result of this paper.

Theorem 5.1. *Suppose assumptions (B.1), (B.2) and (B.3) hold. System (4.1) admits a unique adapted solution in the space.*

$$L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,T;H^1_0(G)\right)\right)\times L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,T;H^1_0(G)\right)\right)\times L^2_{\mathcal{F}}\left(\Omega;L^2\left(0,T;L_Q\right)\right).$$

Proof. When $\alpha=0$, the existence of an adapted solution of system (4.3) is guaranteed by Lemma 4.2. By Lemma 4.4, there exists a $\delta\in(0,1)$, such that for any $\alpha\in[0,\delta]$, system (4.3) admits an adapted solution. From the proof of Lemma 4.4, δ is independent of α. Thus by applying Lemma 4.4 again we see that system (4.3) is solvable for all $\alpha\in[\delta,2\delta]$. Repeating this process, one can show that system (4.3) is solvable for $\alpha=1$. Hence the existence of a solution of system (4.1) is shown.

Now let us prove the uniqueness. Suppose $(\boldsymbol{u},\boldsymbol{v},Z)$ and $(\boldsymbol{u}',\boldsymbol{v}',Z')$ are two solutions. Let $\hat{\boldsymbol{u}}=\boldsymbol{u}-\boldsymbol{u}'$, $\hat{\boldsymbol{v}}=\boldsymbol{v}-\boldsymbol{v}'$, and $\hat{Z}=Z-Z'$. Applying the Itô formula to $\langle\hat{\boldsymbol{u}},\hat{\boldsymbol{v}}\rangle$ to get

$$\begin{aligned}\partial_t\langle\hat{\boldsymbol{u}},\hat{\boldsymbol{v}}\rangle=&\Big\{\left\langle S(t,x,X)-S(t,x,X'),\hat{\boldsymbol{v}}\right\rangle_{H^{-1},H}\\&+\left\langle T(t,x,X)-T(t,x,X'),\hat{\boldsymbol{u}}\right\rangle_{H^{-1},H}\\&+\left\langle\sigma(t,x,X)-\sigma(t,x,X'),\hat{Z}\right\rangle_{L_Q}\Big\}\mathrm{d}t\\&+\left\langle\hat{Z}^*(\hat{\boldsymbol{u}})+\sigma^*(t,x,X)(\hat{\boldsymbol{v}})-\sigma^*(t,x,X')(\hat{\boldsymbol{v}}),\mathrm{d}W\right\rangle.\end{aligned}$$

Thus

$$\begin{aligned}c_2E\left|\hat{\boldsymbol{u}}(T,x)\right|^2&\le E\left\langle\hat{\boldsymbol{u}}(T,x),g\left(\boldsymbol{u}(T,x)\right)-g\left(\boldsymbol{u}'(T,x)\right)\right\rangle\\&\le-c_2E\int_0^T\left(\|\hat{\boldsymbol{u}}\|^2+\|\hat{\boldsymbol{v}}\|^2+\|\hat{Z}\|^2_{L_Q}\right)\mathrm{d}t,\end{aligned}$$

and

$$(\boldsymbol{u},\boldsymbol{v},Z)=(\boldsymbol{u}',\boldsymbol{v}',Z'),\ \text{P-a.s., for every } t\ge0 \text{ and } x\in G.$$

Remark 5.2. *Theorem 5.1 is the abstract counterpart of Theorems 2.4 and 3.1: assumption (B.3) is a one-sided monotonicity condition on the whole triple (S,T,σ) simultaneously, in place of the separate, two-sided conditions (A.2) - (A.3) on (b,σ) and k individually. This is what removes the size restriction on ah (or $\alpha\beta$) seen in (2.1) and (3.2): the continuation method of Lemmas 4.1 - 4.4 only needs local solvability of (4.3) at each step, together with a uniform*

Cauchy estimate, and never needs the forward and backward equations to be solved separately by a single contraction.

6. An Illustrative Example

To illustrate how an FBSPDE of the form (1.2) can arise in an application, consider a stochastic optimal control problem for a reaction-diffusion equation. Let $u(t,x)$ denote the state on a bounded domain G, and suppose that

$$\begin{cases} \partial_t u = a\mathcal{L}u\mathrm{d}t + \left(f(t,x)+\pi(t,x)\right)\mathrm{d}t + \sigma_0 \mathrm{d}W(t,x) \\ u(0,x) = u_0(x),\ t\in[0,T],\ x\in G, \end{cases} \tag{6.1}$$

where f is a given forcing term, π is the control, and $\sigma_0 > 0$ is a constant. Depending on the interpretation of u, this equation can describe, for example, the evolution of a concentration, a population density, or a temperature perturbation under external random forcing.

Suppose that the controller wants to keep the state close to a prescribed target while also penalizing the use of the control. For a quadratic cost functional, the stochastic maximum principle leads to an adjoint backward equation. If $v(t,x)$ denotes the adjoint process and $Z(t,x)$ its martingale part, the optimal control can be expressed in terms of v. The resulting optimality system for (u,v,Z) has the same general forward-backward structure as (1.2); the forward equation contains a feedback term involving the adjoint variable, while the backward equation contains terms depending on the state. Such forward-backward systems are standard in stochastic control; see, for example [2].

In reaction-diffusion control problems, the coupling between the state and adjoint equations can also involve second-order terms. For example, after substituting the optimal control into the state equation, a term involving $\mathcal{L}v$ may appear in the forward equation. This gives a coupling structure similar to that in the linear model (4.2) considered in Lemma 4.2. Such terms are relevant to the monotonicity estimates because they contribute at the $H_0^1(G)$ level rather than only at the $L^2(G)$ level.

This example also gives some interpretation of assumptions (A.1) - (A.3). These conditions impose restrictions on the reaction terms, the diffusivities a and h, and the strength of the feedback between the state and adjoint equations. In particular, the dissipative effects need to be sufficiently strong relative to the coupling terms. Similar connections between reaction-diffusion models and monotonicity conditions for FBSPDEs were discussed in [14].

The purpose of this example is to illustrate the structure of the system rather than to give a specific numerical application. For a particular control problem, the coefficients would have to be checked against (A.1) - (A.3), or against (A.4) in the setting of Section 3, before the corresponding well-posedness theorem can be applied.

7. Concluding Remarks

In this paper, we studied fully coupled FBSPDEs using a monotonicity approach.

Lemma 2.2 and Theorem 2.4 treat the Laplacian-type system (1.2) by a contraction argument, under the coupling condition (2.1) on the diffusivities a and h. Theorem 3.1 extends the argument to general coercive operators under assumption (A.4). Finally, Theorem 5.1, based on the continuation method in Lemmas 4.1 - 4.4, removes the coupling condition at the cost of the stronger one-sided monotonicity assumption (B.3).

These results complement our previous work on FBSPDEs. In particular, they are related to the Lipschitz-based results in [15], the Yosida approximation results in [16] under non-degeneracy, and the weak Galerkin solutions in [17]. Here we obtain existence and uniqueness of adapted solutions of (1.2), (3.1), and (4.1) in L^2-based Sobolev spaces under monotonicity assumptions.

There are several questions that remain open. First, assumptions (A.2) - (A.3) and (B.3) are monotonicity conditions and therefore do not cover coefficients that are only locally Lipschitz or that satisfy a one-sided growth condition without monotonicity. The Yosida approximation in [16] provides a way to treat a broader class of coefficients, but requires additional non-degeneracy and compatibility assumptions. It would be useful to see whether such approximation methods can be combined with the continuation method of Section 5 to weaken the monotonicity assumptions.

Second, the coupling condition (2.1) in Theorems 2.4 and 3.1 remains a restriction of the contraction argument. The example in Section 6 gives some indication of how this condition is related to the relative strength of the diffusion and coupling terms. It would be interesting to determine whether this condition can be weakened or replaced by a sharper condition.

Third, our analysis is carried out in the Hilbert space setting $H_0^1(G) \subset L^2(G) \subset H^{-1}(G)$ with a trace-class Q-Wiener process. It would be natural to consider extensions to degenerate noise, space-time white noise, unbounded domains, and systems with more than two coupled components. FBSPDEs arising in mean-field games with common noise, for example [8] [9], provide one possible direction.

Finally, the results here are qualitative. A numerical method for approximating the adapted solutions would be a useful complement to the well-posedness theory. One possible approach would be to combine Galerkin truncation, as in [17], with a suitable monotone time-discretization scheme. We leave this question for future work.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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