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The Hydrogen 1s Electron as an Event 3 Process: A Gamma Distribution (3, 1) Derived from Bohr Velocity Time-Conversion

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Abstract

The radial probability distribution of the hydrogen 1s orbital is obtained by multiplying the probability density $|\psi|^2$ by the surface-area element of a sphere with radius *r*, $4\pi r^2$. By introducing a characteristic time through the Bohr velocity and interpreting distance as an effective waiting time, the resulting distribution can be mathematically reduced to a Gamma distribution with shape parameter 3. This structure is equivalent to the sum of three independent exponential waiting processes. The origin of the parameter "3" is traced to the factor r^2, which arises from the three-dimensional spherical volume element. Thus, the Gamma-distribution structure is closely related to the dimensionality of space. Based on this observation, we propose a mathematical interpretation in which the electron reaching distance (r) corresponds to the accumulation of probability amplitudes along three independent spatial directions, (x), (y), and (z). In higher-dimensional spaces, the Coulomb potential is modified according to Gauss's law, leading to the well-known "fall-to-the-center" behavior, where stable bound states at the Bohr radius can no longer be maintained. In such cases, the radial distribution no longer reduces to a Gamma distribution. These results suggest that only in three dimensions do the Coulomb potential ($\sim 1/r$), the exponential wave function ($\sim \mathrm{e}^{-r}$), and the spatial volume element ($\sim r^2$) become mutually balanced, giving rise to the Gamma-distribution structure.

Keywords

Wave Function, Poisson Process, Exponential Function, Multiple Universes

1. Introduction

The Schrödinger equation for the 1s orbital of a hydrogen atom is structured based on the Coulomb attraction acting between the central nucleus (positive charge) and a single electron (negative charge). Specifically, it takes the form of the law of conservation of energy, as shown below:

$$\left[-\frac{\hbar^2}{2m}\Delta-\frac{e^2}{4\pi\epsilon_0 r}\right]\psi=E\psi, \tag{1}$$

$\hbar$, Dirac constant; m, mass of the electron; e, elementary charge; ε_0, vacuum permittivity; r, distance from the central nucleus; Ψ, wave function; E, total energy [1]-[5]. When breaking down this structure according to the characteristics of the 1s orbital (ground state), the following three points become essential:

1) The Laplacian operator Δ in the kinetic energy term calculates the "curvature" of the wave function. When attempting to confine an electron to a narrow range (as r decreases), the gradient of the wave function becomes steeper, leading to a sharp increase in kinetic energy. This functions as a "repulsive effect" that prevents the electron from completely collapsing into the nucleus (fall to the center), allowing it to remain stable at a specific distance (near the Bohr radius).

2) The most characteristic feature of the 1s orbital equation is the potential energy term. It takes the shape of a "funnel", where the energy becomes infinitely lower as it approaches the center ($r = 0$). This strong attraction serves to bind the electron around the nucleus and suppress its spread.

Consequently, the Schrödinger equation for the 1s orbital is structured to find the exact point where the "pull toward the center (Coulomb force)" and the "resistance to confinement (kinetic energy of the wave)" reach an equilibrium. The extent of the 1s orbital (radial distribution) that we observe is determined as a result of this balance.

3) Since the 1s orbital is spherically symmetric, the angular components (θ,φ) of the three-dimensional Laplacian operator can be neglected, leaving a structure that depends solely on changes in the radial direction r. As a result, the solution to the equation takes the form of a simple exponential function $\psi(r)=C\mathrm{e}^{-r/a_0}$; a_0, Bohr radius, leading to exponential decay.

From these three points, it can be concluded that the Schrödinger equation for the 1s orbital is structured to find the exact point of equilibrium between the "inward pull toward the center" (Coulomb force) and the "resistance to confinement" (kinetic energy of the wave). The spatial extent of the 1s orbital—the radial distribution we observe—is determined as the direct result of this physical balance. This balance is mathematically fragile; in an n-dimensional space, we assume a generalized Coulomb interaction derived from the n-dimensional Gauss law [6]. The total energy behaves as:

$$E(r)\approx\frac{A}{r^2}-\frac{B}{r^{n-2}}. \tag{2}$$

In our three-dimensional world ($n = 3$), the kinetic term ($1/r^2$) dominates at short distances over the potential term ($1/r$), creating a stable minimum. However, in four or more dimensions ($n \geq 4$), the potential term decreases as fast as or faster than the kinetic term. This causes the energy to plummet to negative infinity as $r \to 0$, making a stable Bohr radius impossible. The position of an electron in the 1s orbital can only be described probabilistically due to the uncertainty principle.

The objective of this study is to examine, from a probabilistic perspective, why three-dimensional space plays a special role in the hydrogen atom. By introducing a characteristic time scale based on the Bohr velocity, the radial coordinate can be reinterpreted in temporal form, and the hydrogenic 1s radial probability distribution reduces to a Gamma distribution with parameters (3, 1). Although the emergence of a Gamma/Erlang form after an appropriate rescaling of the hydrogen 1s radial distribution is mathematically straightforward and implicitly contained in standard quantum mechanics [7], its physical and probabilistic interpretation has not been fully explored. The novelty of the present work lies not in the mathematical reduction itself, but in the reinterpretation of the resulting Gamma distribution as a multi-stage waiting-time process associated with the three-dimensional spatial structure of a Coulomb-bound electron. In this framework, the shape parameter 3 is interpreted as reflecting the geometric degrees of freedom of three-dimensional space.

2. Deriving a Gamma Distribution by Nondimensionalizing Time

The probability that an electron in the 1s orbital of a hydrogen atom is found at a distance r from the nucleus is expressed as follows, taking into account the surface area of the spherical shell.

$$P(r) = 4\pi r^2 \left|\psi(r)\right|^2 = \frac{4}{a_0^{\,3}} r^2 \mathrm{e}^{-2r/a_0}. \tag{3}$$

Accordingly, in order to treat the distance as a time variable, we define the time using a characteristic velocity of the electron as follows.

$$t \equiv \frac{r}{v} = \frac{r}{\alpha c};\ \alpha, \text{fine-structure constant};\ c, \text{speed of light}. \tag{4}$$

Here, the characteristic velocity of the electron in the 1s orbital is taken to be $v \sim \alpha c$, as suggested by the Bohr model. In this study, the transformation $t = r/(\alpha c)$ is introduced primarily as a formal probabilistic change of variables based on the characteristic Bohr velocity. The resulting temporal representation should therefore be interpreted as a mathematical and statistical analogy, rather than as a directly measurable electron arrival-time observable or a classical trajectory description. We define the time t_B as follows.

$$t_B \equiv \frac{a_0}{\alpha c} = \frac{1}{\alpha c} \cdot \frac{\hbar}{mc\alpha} = \frac{\hbar}{mc^2\alpha^2}. \tag{5}$$

Substituting $r = \alpha ct$ and $\mathrm{d}r = \alpha c\mathrm{d}t$ into $P(r)\mathrm{d}r$ (which denotes the probability that the electron is found within the spherical shell between r and $r+\mathrm{d}r$), we obtain

$$P_t(t)\,\mathrm{d}t = \frac{4(\alpha c)^3}{a_0^3} t^2 \mathrm{e}^{-2t/t_B}\,\mathrm{d}t. \tag{6}$$

From (5),

$$P_t(t)\,\mathrm{d}t = \frac{4}{t_B^3} t^2 \mathrm{e}^{-2t/t_B}\,\mathrm{d}t. \tag{7}$$

Therefore, the distribution in time can be expressed as follows.

$$P_t(t) = \frac{4}{t_B^3} t^2 \mathrm{e}^{-2t/t_B}. \tag{8}$$

3. Reduction of the Radial Distribution Function to a Gamma Distribution $\Gamma(3,1)$

As shown below, by further introducing a dimensionless time variable u, a Gamma distribution can be derived.

$$u \equiv \frac{2t}{t_B}. \tag{10}$$

$$t = \frac{t_B}{2}u \text{ and } \mathrm{d}t = \frac{t_B}{2}\mathrm{d}u. \tag{11}$$

Substituting these terms into radial distribution function (8),

$$P_t(t)\,\mathrm{d}t = \frac{4}{t_B^3}\left(\frac{t_B}{2}u\right)^2 \mathrm{e}^{-u}\left(\frac{t_B}{2}\mathrm{d}u\right) = \frac{1}{2}u^2\mathrm{e}^{-u}\mathrm{d}u. \tag{12}$$

Thus,

$$P_t(t) = \frac{1}{\Gamma(3)}u^2\mathrm{e}^{-u} \Rightarrow P_u(u) \equiv \frac{1}{\Gamma(3)}u^2\mathrm{e}^{-u} \Rightarrow u \equiv \frac{2t}{t_B} \sim \Gamma(3,1). \tag{13}$$

$$\langle u \rangle = 3 \text{ and } Var(u) = 3 \tag{14}$$

From (14), the average "arrival time" is given as follows:

$$\left\langle \frac{2t}{t_B} \right\rangle = 3 \Rightarrow \langle t \rangle = \frac{3}{2}t_B = \frac{3}{2}\frac{\hbar}{mc^2\alpha^2}. \tag{15}$$

$$Var\left(\frac{2t}{t_B}\right) = 3 \Rightarrow Var(t) = 3\left(\frac{t_B}{2}\right)^2 = \frac{3}{4}\left(\frac{\hbar}{mc^2\alpha^2}\right)^2. \tag{16}$$

4. Mathematical Interpretation of the Gamma Distribution $\Gamma(3,\lambda)$

$\Gamma(3,\lambda)$ can be expressed as the sum

$$S_3 = X_1 + X_2 + X_3;\; X_i \sim \exp(\lambda), \tag{17}$$

where X_1, X_2, and X_3, are independent random variables that follow an exponential distribution with shape parameter 1 (rate λ). Although the exponential random variable is memoryless, we show below that the sum of three independent exponential random variables does not possess the memoryless property. We first demonstrate this for the case of two random variables. The probability density function of $S_2 = X_1 + X_2$ can be obtained by the convolution of the corresponding distributions:

$$\begin{aligned} f_{S_2}(t) &= \int_0^t f_{X_1}(x) f_{X_2}(t-x)\,\mathrm{d}x = \int_0^t \lambda \mathrm{e}^{-\lambda x} \lambda \mathrm{e}^{-\lambda(t-x)}\,\mathrm{d}x \\ &= \lambda^2 \mathrm{e}^{-\lambda t} \int_0^t \mathrm{d}x = \lambda^2 t \mathrm{e}^{-\lambda t}. \end{aligned} \quad (18)$$

This is $\Gamma(2,\lambda)$. Similarly, for $n = 3$, $\Gamma(3,\lambda)$ is obtained by the following calculation.

$$S_3 = S_2 + X_3. \quad (19)$$

$$\begin{aligned} f_{S_3}(t) &= \int_0^t f_{S_2}(x) f_{X_3}(t-x)\,\mathrm{d}x = \int_0^t \lambda^2 x \mathrm{e}^{-\lambda x} \lambda \mathrm{e}^{-\lambda(t-x)}\,\mathrm{d}x \\ &= \lambda^3 \mathrm{e}^{-\lambda t} \int_0^t x\,\mathrm{d}x = \lambda^3 \mathrm{e}^{-\lambda t} \frac{t^2}{2} = \frac{\lambda^3 t^2}{\Gamma(3)} \mathrm{e}^{-\lambda t}. \end{aligned} \quad (20)$$

We show that $\Gamma(3,\lambda)$ is not memoryless using the hazard function as follows. This probability density function is

$$f_{S_3}(t) = \frac{\lambda^3 t^2}{\Gamma(3)} \mathrm{e}^{-\lambda t}. \quad (21)$$

The survival function $S_3(t)$ is given by

$$S_3(t) = 1 - F_{S_3}(t) = \mathrm{e}^{-\lambda t}\left(1 + \lambda t + \frac{(\lambda t)^2}{2}\right). \quad (22)$$

From the above, the hazard function is given by

$$h(t) = \frac{f_{S_3}(t)}{S_3(t)} = \frac{\lambda^3 t^2}{2\left(1 + \lambda t + \frac{(\lambda t)^2}{2}\right)}. \quad (23)$$

$$h'(t) = \frac{2\lambda^3 t(2 + \lambda t)}{\left(2 + 2\lambda t + (\lambda t)^2\right)^2}. \quad (24)$$

As $\lambda > 0$ and $t > 0$, we obtain $h'(t) > 0$. Unlike the exponential distribution, the hazard function of $\Gamma(3,\lambda)$ is not constant, but increases monotonically and converges to λ as $t \to \infty$. Therefore, $\Gamma(3,\lambda)$ does not possess the memoryless property. When the three independent random variables are interpreted as the waiting times until three successive events occur, the increasing hazard function indicates that the occurrence of the next event becomes progressively more likely as time elapses. Accordingly, S_3 can be mathematically interpreted as the sum of the waiting times associated with three independent events. This interpretation implies that $\Gamma(3,\lambda)$ possesses an intrinsic multistage structure. In state A, corresponding to the initial stage, three stages remain to be completed. In contrast, in state B, where two stages have already been completed, only one additional stage remains. Consequently, the distribution of the remaining waiting time differs substantially between states A and B. This dependence on the current stage reflects the non-memoryless nature of $\Gamma(3,\lambda)$. Thus, the distribution may be viewed as consisting of three independent stages that must be traversed before a given waiting time is reached. The shape parameter (3) should therefore be interpreted as the number of independent stages (or events), rather than as degrees of freedom, except in special cases such as the chi-square distribution.

5. Mathematical Interpretation of Three Independent Events in Terms of Waiting Times

As a hypothesis, for an electron to reach a distance r from the nucleus, the probability amplitudes must accumulate along three independent directions. Interpreting this as being achieved over time, the corresponding waiting time can be expressed as follows. $\Gamma(3,1)$ can be expressed as the sum

$$T = T_1 + T_2 + T_3; T_i \sim \exp(1), \tag{25}$$

where T_1, T_2, and T_3, are independent random variables of waiting time that follow an exponential distribution with shape parameter 1 (rate 1) (**Figure 1**).

From (15),

$$\langle t \rangle = \frac{3}{2} t_B = \frac{3}{2} \frac{\hbar}{mc^2 \alpha^2} \Rightarrow \langle t_i \rangle = \frac{1}{2} t_B = \frac{1}{2} \frac{\hbar}{mc^2 \alpha^2}. \tag{26}$$

From (16),

$$Var(t) = 3\left(\frac{t_B}{2}\right)^2 = \frac{3}{4}\left(\frac{\hbar}{mc^2 \alpha^2}\right)^2 \Rightarrow Var(t_i) = \left(\frac{t_B}{2}\right)^2 = \frac{1}{4}\left(\frac{\hbar}{mc^2 \alpha^2}\right)^2. \tag{27}$$

It is found that the time scales of the mean and variance of the waiting time in each direction are governed by the fine-structure constant. In particular, since the variance in each direction is finite, the waiting time exhibits a finite spread. When this is translated into distance, it implies that the electron's position is spatially extended.

6. For Example, the Gamma Distribution Does Not Arise in Four Dimensions

In three dimensions, the potential is given by

$$V(r) \propto -\frac{1}{r}. \tag{28}$$

The 1s wave function is given by

$$\psi(r) \propto e^{-r/a_0}. \tag{29}$$

Thus,

$$P_3(r) \propto r^2 e^{-2r/a_0}. \tag{30}$$

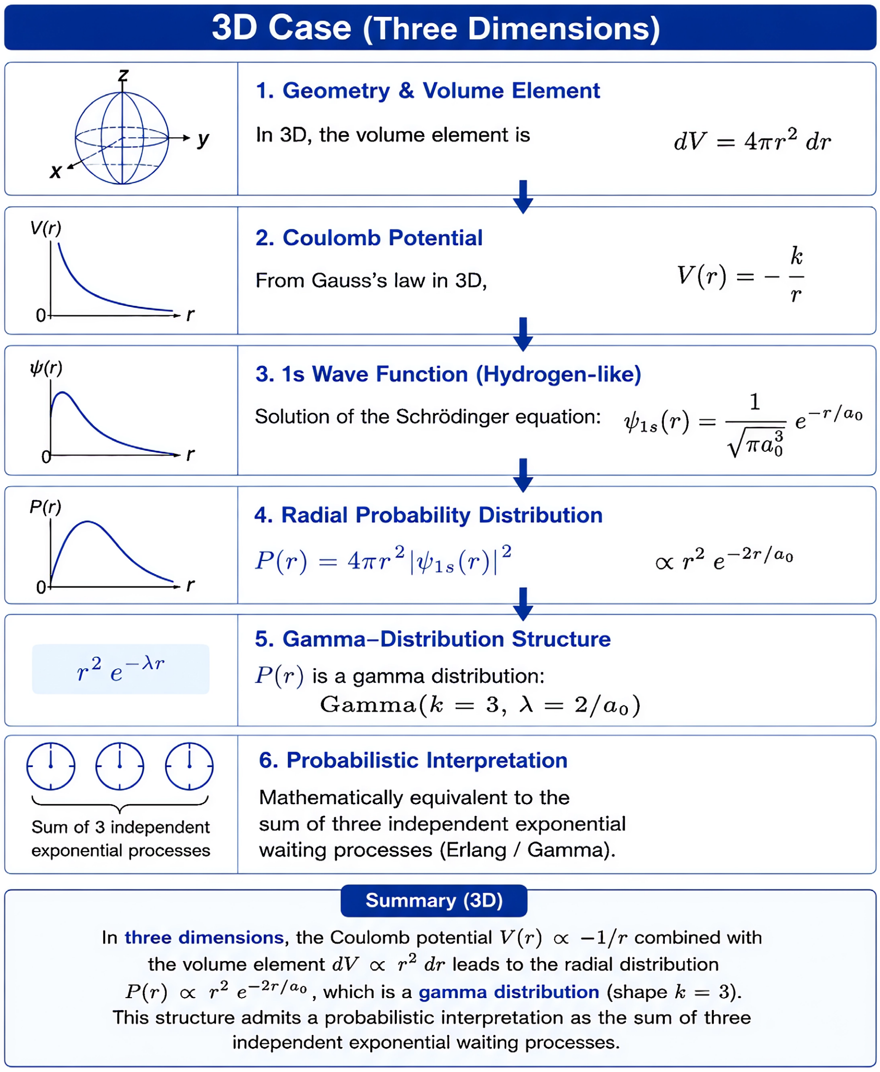

Figure 1. Overall structure for the hydrogen 1s state in three dimensions.

This corresponds to a gamma distribution $\Gamma\left(3, \frac{2}{a_0}\right)$. In four dimensions, the radial Schrödinger equation for a spherically symmetric state is expressed as follows:

$$R''(r) + \frac{3}{r}R'(r) + \left(\frac{A}{r^2} - \kappa^2\right)R(r) = 0;\ \kappa = \frac{\sqrt{-2mE}}{\hbar},\ A = \frac{2mC}{\hbar^2} - \ell(\ell+2). \quad (31)$$

C is the strength of the potential, and ℓ denotes the orbital angular momentum quantum number in quantum mechanics, and $\ell = 0$ for an s orbital. The general form of the solution is given as follows:

$$R(r) \propto \frac{1}{r} K_\nu(\kappa r), \quad (32)$$

where K_ν denotes the modified Bessel function of the second kind. Therefore, the radial distribution in four dimensions is given as follows:

$$P_4(r) \propto r^3 |R(r)|^2 \propto rK_\nu(\kappa r)^2. \quad (33)$$

This is not of the form of a gamma distribution. In particular, in the asymptotic region, it becomes

$$K_\nu(\kappa r) \sim \sqrt{\frac{\pi}{2\kappa r}} e^{-\kappa r} \Rightarrow P_4(r) \propto e^{-2\kappa r}. \quad (34)$$

Therefore, it is found that this does not take the following form of a gamma distribution:

$$r^3 e^{-\lambda r}. \quad (35)$$

Similarly, in five or higher dimensions, the Coulomb potential itself changes according to Gauss's law, and the radial distribution function is no longer described by a gamma distribution (Figure 2). Therefore, it is suggested that only in three dimensions do the Coulomb potential and the spatial volume element become specially compatible, giving rise to a gamma-distribution structure.

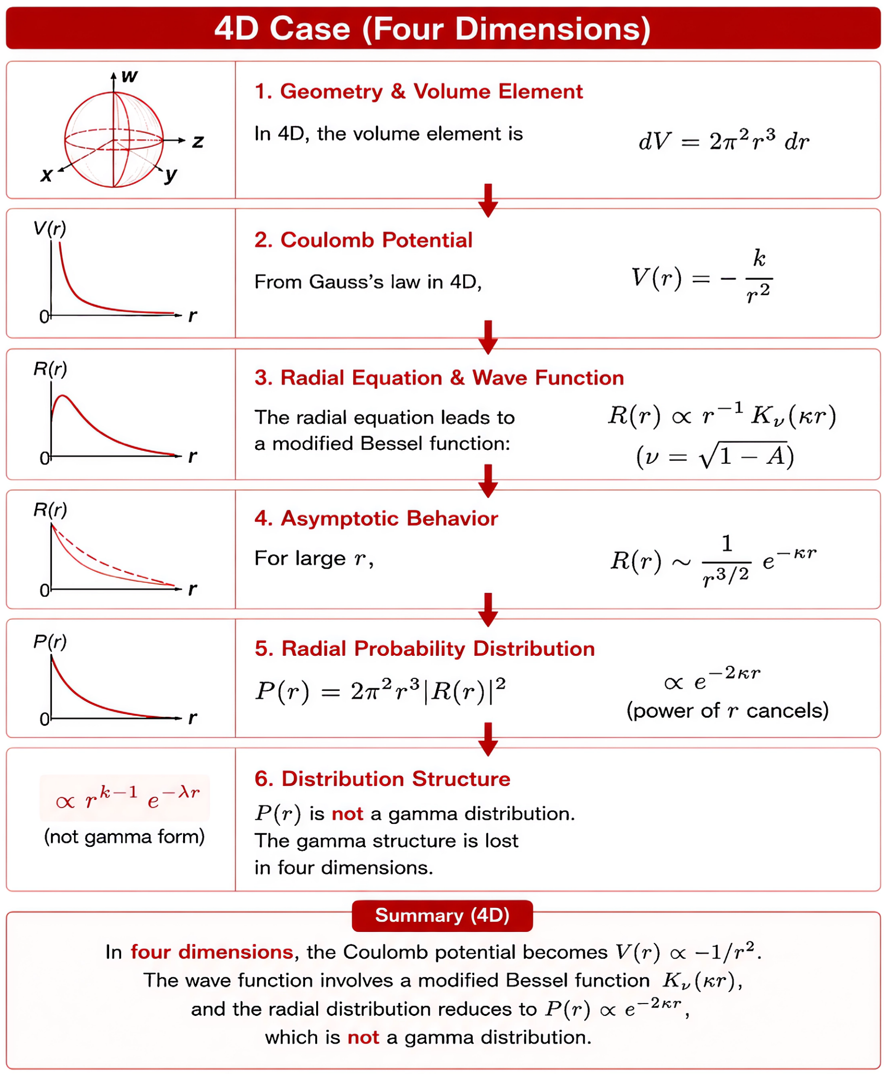

Figure 2. Overall structure for the hydrogen 1s state in four dimensions.

7. Discussion

In the present probabilistic interpretation, the shape parameter 3 may be viewed as reflecting the geometric structure of three-dimensional spherical space through the radial volume element r^2. In addition, the finite variance of the Γ (3, 1) waiting-time distribution implies that the electron is not localized at a single radius, but instead possesses a finite spatial spread around the nucleus. This interpretation is fully consistent with the wave-mechanical description of the electron provided by the Schrödinger equation, rather than with the classical picture of a point particle following a deterministic trajectory. Consequently, the probabilistic structure associated with the Gamma distribution provides an alternative statistical interpretation of the spatial delocalization inherent in quantum mechanics. In this sense, the present analysis offers a probabilistic perspective consistent with both the uncertainty principle and the wave nature of the electron in the hydrogen atom.

In multiverse or higher-dimensional brane theories, it is often assumed that a three-dimensional brane is embedded within a higher-dimensional bulk space [8]. As discussed in the Introduction, even in the simple case of the hydrogen atom, stable bound electronic structures can exist only when the Coulomb interaction is effectively confined to three-dimensional space. In higher-dimensional spaces, the Coulomb potential becomes increasingly singular, leading to the phenomenon known as "fall to the center," such that stable electron clouds cannot be maintained around the nucleus. Remarkably, in three-dimensional space, the Coulomb interaction ($\sim 1/r$), the exponential form of the Schrödinger wave function ($\sim \mathrm{e}^{-r}$), and the spherical volume element ($\sim r^2$) combine in a highly balanced manner to produce the Γ (3, 1) distribution structure. When this distribution is interpreted as the sum of three independent exponential waiting processes, the characteristic time scales governing the mean and variance of each component are found to be related to the fine-structure constant through the Bohr scale. Although the precise physical meaning of these independent exponentially distributed variables remains unclear, they may be mathematically interpreted as effective waiting processes associated with the three spatial directions, since the shape parameter itself originates from spatial dimensionality.

In tokamak-type magnetic confinement fusion devices, strong magnetic fields constrain charged particles to undergo Larmor motion, effectively reducing transverse electron motion [9]. In one of our previous studies [10], we speculatively suggested that under such conditions the electron probability distribution might become more localized around the nucleus. Within the present probabilistic framework, this effect may be qualitatively interpreted as a perturbation of the three independent waiting-time processes associated with the Gamma-distribution representation of the hydrogenic 1s state. From this viewpoint, the formation of the total waiting time corresponding to the Bohr radius could be modified, potentially leading to a tendency toward stronger localization near the nucleus. Although this interpretation remains speculative and does not constitute a rigorous dynamical derivation, it may provide an intuitive probabilistic perspective on possible suppression mechanisms of ionization and plasma formation under strong magnetic confinement.

8. Conclusion

In this study, the radial probability distribution of the hydrogenic 1s orbital was reexamined from a probabilistic perspective. By introducing a characteristic temporal scaling based on the Bohr velocity, the radial distribution was reformulated into a Gamma distribution of the form Γ (3, 1). Although the mathematical reduction itself is straightforward, the present work focused on its probabilistic interpretation as a multi-stage waiting-time structure associated with the geometric properties of three-dimensional spherical space. The analysis showed that the shape parameter 3 is mathematically connected to the radial volume element r^2 appearing in three-dimensional spherical coordinates. In this framework, the hydrogenic radial distribution may be viewed as reflecting an underlying probabilistic structure related to the spatial geometry of the Coulomb-bound electron. The corresponding hazard function further indicates that the process is not memoryless, unlike a simple exponential distribution. The present interpretation is intended as a formal probabilistic analogy rather than a direct physical derivation of electron dynamics or measurable arrival times. Nevertheless, the Gamma-distribution viewpoint may provide an alternative conceptual framework for understanding geometric aspects of hydrogenic quantum states and their possible extensions to higher-dimensional or constrained systems.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Ozawa, M. (2003) Universally Valid Reformulation of the Heisenberg Uncertainty Principle on Noise and Disturbance in Measurement. *Physical Review A*, **67**, Article 042105. https://doi.org/10.1103/physreva.67.042105

[2] Kurai, T. (2023) A New Electron Charge Distribution Function of Electron. *Journal of Modern Physics*, **14**, 111-126. https://doi.org/10.4236/jmp.2023.142008

[3] Girone, G., Massari, A., Manca, F. and Marin, C. (2020) Mean Difference of Truncated Normal Distribution. *Applied Mathematics*, **11**, 1162-1166. https://doi.org/10.4236/am.2020.1111078

[4] Wei, Z., Peng, T. and Zhou, X. (2020) The Alpha-Beta-Gamma Skew Normal Distribution and Its Application. *Open Journal of Statistics*, **10**, 1057-1071. https://doi.org/10.4236/ojs.2020.106060

[5] Hey, T. and Walters, P. (2009) The New Quantum Universe. Cambridge University Press.

[6] Ehrenfest, P. (1917) In What Way Does It Become Manifest in the Fundamental Laws of Physics that Space Has Three Dimensions? *Proceedings of the Royal Netherlands Academy of Arts and Sciences*, *Amsterdam*, **20**, 200-209.

[7] Bethe, H.A. and Salpeter, E.E. (1977) Quantum Mechanics of One- and Two-Electron

Atoms. Plenum Press. https://doi.org/10.1007/978-1-4613-4104-8

[8] Osaka, M. (2026) A Mathematical Exploration of the Correlation between the Speeds of Light in Adjacent Universes. *Applied Mathematics*, **17**, 164-174. https://doi.org/10.4236/am.2026.173010

[9] Clery, D. (2014) A Piece of the Sun: The Quest for Fusion Theory. MIT Press.

[10] Osaka, M. (2024) Approximating the Radial Distribution Function of the Electron in a Hydrogen Atom by a Normal Distribution Suggests That Magnetic Confinement Fusion Would Be Less Energy Efficient than Inertial Confinement Fusion. *Applied Mathematics*, **15**, 585-593. https://doi.org/10.4236/am.2024.159035