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Reproducing Kernel Hilbert Space Technique for Fourth-Order Singular Boundary Value Problems

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Abstract

Reproducing kernel Hilbert space method is utilized in this paper as an efficient approach to solve singular fourth order boundary value problems of mixed form Fredholm-Volterra integro-differential equations. In order to obtain the required nodal values, the algorithm developed using two smooth reproducing kernel functions. The solution philosophy is based on applying the Gram-Schmidt process to the kernel function obtained in the space $W_2^5[0,1]$ to produce an orthogonal basis. From that point onward, the orthogonal basis was built for the purpose of formulating and utilizing numerical solutions in the same space. Some linear and nonlinear numerical issues were broken down to delineate the technique and affirm the execution of the proposed strategy. The numerical outcomes emphasize the method role in improving the initial approximation, handling boundary conditions, and refining the solution throughout the iterative process to shed light on such singular equations.

Keywords

Hilbert Space, Integro-Differential Equation, Gram-Schmidt Process, Fredholm-Volterra Integral Equations, Reproducing Kernel Function

1. Introduction

Numerous problems in various fields of science can be diminished, by applying some proper discretization into integro-differential equations (IDEs). In fact, numerous scientific details of engineering applications, power generators, physical phenomena such as fluid mechanics, biological and chemical models contain IDEs with boundary conditions (BCs) etc. [1]-[5]. The term "singular" is used to emphasize that the coefficients of the considered equation may take value zero at one or more point in the given domain. Anyhow, traditional numerical methods such as finite deference, homotropy analysis and shooting technique encountered loss of accuracy and significant difficulties or may loss the convergence when they applied to problem with singularities. Since it is normally difficult to get the closed frame.

The technique of reproducing kernel has been employed to solve differential equations since the mid-20th century as a novel solver for the BVPs to provide solutions in a structured and systematic manner. In 1907, Zaremba [6] was the first to introduce the kernel corresponding to a class of functions without giving any name or developing any theory, and he was also the first to mention its reproducible property. In 1909, Mercer [7] developed the theory of continuous positive definite kernels under the name of the theory of "positive definite kernels". He studied the relationship between positive and negative type functions that satisfy reproducibility property with the theory of integral equations. Moreover, he demonstrated some qualitative characteristics of these positive definite kernels that were not present in all other continuous kernels of IEs.

The goal of numerical techniques is to provide precise strategies to deal with problem in a numerical frame. This technique relies on the use of high-precision computers to perform the steps in order to obtain the result from the raw data.

The RKHS method is a numerical technique developed to solve substantial assortment of ordinary and partial differential equations. The general theory of RKs was systematized in 1650 by Aronszajn [8]. He summarized the previous related research results and used "RK functions" as the identical term for these different functions, so the foundations of RK theory were set up. The applications of the RKHS method in solving different categories of equations with some characteristics and advantages can be found in [9]-[21]. Advances in computer hardware and software make verification and simulation easier, so they are necessary in applied sciences. However, the RKHS method was intentionally examined to solve numerically for the following fourth-order singular BVPs of the mixed-type Fredholm-Volterra IDE in the unknown function u:

$$p_4(x)u^{(4)} + p_3(x)u^{(3)} + p_2(x)u^{(2)} + p_1(x)u^{(1)} = f(x,u) + T[u](x) \tag{1}$$

In which $T[u](x)$ operator is given as

$$T[u](x) = \lambda_1 \int_a^b k_1(x,t) J_1(u(t))\,\mathrm{d}t + \int_a^x k_2(x,t) J_2(u(t))\,\mathrm{d}t \tag{2}$$

With respect to the following set of constraint BCs:

$$u(a) = \alpha_1, u(b) = \beta_2 \text{ and } u'(a) = \alpha_2, u'(b) = \beta_2 \tag{3}$$

$$u(a) = \alpha_1, u(b) = \beta_2 \text{ and } u''(a) = \alpha_2, u''(b) = \beta_2$$

where $x \in [a,b]$, the function $f(x,u)$ is a continuous real valued function with two variables, λ_1, λ_2 are constant parameters, $k_1(x,t), k_2(x,t)$ are two known kernel functions of two variables x and t, $J_1(u(t)), J_2(u(t))$ are two known functions of one variable, and $T[u](x)$ is Fredholm- Volterra mixed type operator.

Discussion about solvability of singular BVPs of Fredholm-Volterra mixed type operator subject to different separated BCs is scarce and missing. Lately, some researchers have investigated the numerical applicability of IDEs in the case of nonsingular coefficients by using some of the well-known methods. These methods and their properties can be found in [22]-[26].

Most physical and engineering problems are often described in terms of differential equations, integral equations, or IDEs with conditions imposed at one point or more points see A.Majid Wazwaz book [27].

BVPs of singular functions form emerge as often as possible in numerous branches of science and engineering, such as fluid mechanics, atomic calculations, biological systems, chemical models and in the study of nonlinear elliptic equations see references [28] [29].

In this paper, we present a novel application of the reproducing kernel Hilbert space (RKHS) method to solve linear and nonlinear fourth-order singular BVPs of the mixed-type Fredholm-Volterra integro-differential equations

2. Preliminaries

Definition 2.1. The inner product and the norm of the space $W_2^1[0,1]=\{z(x): z$ is real-valued function which is absolutely continuous on $[0,1]$ and $z' \in L^2[0,1]$ are defined respectively as

$$\langle z_1(x), z_2(x)\rangle_{W_2^1} = z_1(0)z_2(0)+\int_0^1 z_1'(x)z_2'(x)\mathrm{d}x$$

$$\|z\|_{W_2^1} = \sqrt{\langle z(x), z(x)\rangle_{W_2^1}}$$

where $z_1, z_2 \in W_2^1[0,1]$.

Theorem 2.1. The space $W_2^1[0,1]$ is a RKHS. Using Mathematica software, it was found that its reproduction kernel function $R_x(y)$ is

$$R_x(y)=\begin{cases} R_{1x}(y)=1+y, & y\le x, \\ R_{2x}(y)=1+x, & y>x, \end{cases}$$

Definition 2.2. The inner product and the norm of the space $W_2^5[0,1]=\{z(x): z^{(i)}, i=0,1,2,3,4\}$ are real-valued functions that are absolutely continuous on $[0,1]$, $z^{(5)} \in L^2[0,1]$, and $z(0)=z'(0)=z(1)=z'(1)=0\}$. are defined respectively as

$$\langle z_1(x), z_2(x)\rangle_{W_2^5} = \sum_{i=0}^{2} z_1^{(i)}(0)z_2^{(i)}(0)+\sum_{i=0}^{1} z_1^{(i)}(1)z_2^{(i)}(1)+\int_0^1 z_1^{(5)}(x)z_2^{(5)}(x)\mathrm{d}x$$

$$\|z\|_{W_2^5} = \sqrt{\langle z(x), z(x)\rangle_{W_2^5}}$$

where $z_1, z_2 \in W_2^5[0,1]$.

Theorem 2.2. The space $W_2^5[0,1]$ is a RKHS. Using Mathematica, it was found that its reproduction kernel function $K_x(y)$ is

$$K_x(y)=\begin{cases} K_{1x}(y)=\sum_{i=0}^{9} p_i(x)y^i, y\le x,\\ K_{2x}(y)=\sum_{i=0}^{9} q_i(x)y^i, y> x,\end{cases}$$

where

$$\begin{aligned} K_{1x}(y)=\frac{1}{362880}(x-1)^2 y^2\times\big(&5x^3(8-15y)y+x^6(19-12y)y+y^7\\ &+xy^6(-9+2y)+3x^5y(-10+3y)+x^7y(-4+3y)+5x^4y(1+6y)\\ &+3x^2\left(30240-60480y+30240y^2+12y^5-6y^6+y^7\right)\big)\end{aligned}$$

$$\begin{aligned} K_{2x}(y)=\frac{1}{362880}x^2(y-1)^2\times\big(&90720y^2+36x^5y^2-9x^6y(1+2y)\\ &+x^7\left(1+2y+3y^2\right)+xy^2\left(-181440+40y+5y^2-30y^3+19y^4-4y^5\right)\\ &+3x^2y^2\left(30240-25y+10y^2+3y^3-4y^4+y^5\right)\big)\end{aligned}$$

3. Problem Formulation

The key aspect of the process is selecting appropriate linear operator depending on the inner product spaces $W_2^5[0,1]$ and $W_2^1[0,1]$, this part is the problem formulating. The nonhomogenous BC's must be converted using appropriate transformation to homogeneous BC's in order to apply the RKHS method. Denote the solution of the new equation by $u(x)$. we get

$$p_4(x)u^{(4)}+p_3(x)u^{(3)}+p_2(x)u^{(2)}+p_1(x)u^{(1)}=f(x,u)+T[u](x)$$

where $T[u](x)$ is given by

$$T[u](x)=\lambda_1\int_0^1 k_1(x,t)J_1(u(t))\mathrm{d}t+\lambda_2\int_0^x k_2(x,t)J_2(u(t))\mathrm{d}t$$

subject to the following constraint boundary conditions:

$$\begin{aligned} &u(0)=u'(0)=0,\\ &u(1)=u'(1)=0.\end{aligned}$$

To perform the procedure, we introduce the operator

$$L:W_2^5[0,1]\to W_2^1[0,1] \tag{4}$$

as

$$Lu(x)=p_4(x)u^{(4)}+p_3(x)u^{(3)}+p_2(x)u^{(2)}+p_1(x)u^{(1)} \tag{5}$$

Thus, discretized form of equivalent equations can be obtained as follows:

$$Lu(x)=F(x,u(x),T[u](x)) \tag{6}$$

subject to the boundary conditions

$$\begin{aligned} &u(0)=u'(0)=0,\\ &u(1)=u'(1)=0.\end{aligned} \tag{7}$$

in which the function F is constructed as followes

$$F\left(x,u(x),T[u](x)\right)=f(x,u)+T[u](x) \tag{8}$$

Theorem 3.1. The operator $L:W_2^5[0,1]\to W_2^1[0,1]$ is bounded and linear.

Proof. The lineart part is obvious. From definition 2.2. we have

$$\left\|(Lu)(x)\right\|_{W_2^1}^2=\left[(Lu)(0)\right]^2+\int_0^1\left[(Lu)'(x)\right]^2\,\mathrm{d}x \tag{9}$$

Using the reproducing kernel function properties we can write

$$v(x)=\left\langle v(y),K_x(y)\right\rangle_{W_2^5}$$

$$(Lv)(x)=\left\langle v(y),(LK_x)(y)\right\rangle_{W_2^5}$$

$$(Lv)'(x)=\left\langle v(y),(LK_x)'(y)\right\rangle_{W_2^5}$$

Schwarz inequality gives

$$\begin{aligned}\left|(Lu)(x)\right|&=\left|\left\langle u(x),(LK_x)(x)\right\rangle_{W_2^5}\right|\\&\le\left\|LK_x(x)\right\|_{W_2^5}\left\|u(x)\right\|_{W_2^5}\\&=M_1\left\|u(x)\right\|_{W_2^5}\end{aligned}$$

$$\begin{aligned}\left|(Lu)'(x)\right|&=\left|\left\langle v(x),(LK_x)'(x)\right\rangle_{W_2^5}\right|\\&\le\left\|(LK_x)'(x)\right\|_{W_2^5}\left\|u(x)\right\|_{W_2^5}\\&=M_2\left\|u(x)\right\|_{W_2^5}\end{aligned}$$

Thus, after simple compuations, we get

$$\begin{aligned}\left\|(Lu)(x)\right\|_{W_2^1}^2&=\left[(Lu)(0)\right]^2+\int_0^1\left[(Lu)'(x)\right]^2\,\mathrm{d}x\\&\le\left(M_1^2+M_2^2\right)\left\|u(x)\right\|_{W_2^5}^2\end{aligned}$$

or $\left\|(Lv)(x)\right\|_{W_2^1}\le M\left\|v(x)\right\|_{W_2^5}$, where $M=\sqrt{M_1^2+M_2^2}$.

Theorem 3.2. The sequense $\left\{\psi_i(x)\right\}_{i=1}^{\infty}$ is a complete function system for the space $W_2^5[0,1]$ and

$$\psi_i(x)=L_yK_x(y)\Big|_{y=x_i}.$$

Proof.

$$\begin{aligned}\psi_i(x)&=L_i^*\varphi(x)\\&=\left\langle L_i^*\varphi(y),K_x(y)\right\rangle_{W_2^5}\\&=\left\langle \varphi_i(y),L_yK_x(y)\right\rangle_{W_2^1}\\&=L_yK_x(y)\Big|_{y=x_i}.\end{aligned}$$

For each $u(x)\in W_2^5[0,1]$, let $\left\langle u(x),\psi_i(x)\right\rangle_{W_2^5}=0$, $\forall i\in\mathbb{N}$. Then,

$$\langle u(x),\psi_i(x)\rangle_{W_2^5}=\langle u(x),L^*\varphi_i(x)\rangle_{W_2^5}=\langle Lu(x),\varphi_i(x)\rangle_{W_2^1}=Lu(x_i)=0 \quad (10)$$

Therefore $Lu(x)=0$. Since L has a trivial kernel under the given boundary conditions, $u(x)=0$. Using Gram-Schmidt orthogonalization process the series $\{\bar{\psi}_i(x)\}_{i=1}^{\infty}$ is given by

$$\bar{\psi}_i(x)=\sum_{k=1}^{i}\beta_{ik}\psi_k(x) \quad (11)$$

where

$$\beta_{ij}=\begin{cases}\dfrac{1}{\|\psi_1\|_{W_2^5}} & i=j=1\\ \dfrac{1}{d_{ik}} & i=j\neq 1\\ -\dfrac{1}{d_{ik}}\displaystyle\sum_{k=j}^{i-1}c_{ik}\beta_{kj} & i>j\end{cases}$$

where $d_{ik}=\sqrt{\|\psi_i\|_{W_2^5}^2-\sum_{k=1}^{i-1}c_{ik}^2}$ and $c_{ik}=\langle\psi_i,\bar{\psi}_k\rangle_{W_2^5}$.

Lemma 3.1. For each $i=0,1,2,3,4$ there exist a positive M_i such that $\|u^{(i)}(x)\|_C\leq M_i\|u(x)\|_{W_2^5}$, where

$$\|u(x)\|_C=\max_{0\leq x\leq 1}|u(x)|$$

Proof. $\forall x\in[0,1]$, we have

$$u^{(i)}(x)=\langle u(y),K_x^{(i)}(y)\rangle_{W_2^5},\ i=0,1,2,3,4$$

By the reproduction kernel function $K_x(y)$ given in theorem 2.2 it follows that for each $i=0,1,2,3,4$

$$\|K_x^{(i)}(y)\|_{W_2^5}\leq M_i$$

Thus,

$$\begin{aligned}|u^{(i)}(x)|&=\left|\langle u(x),K_x^{(i)}(x)\rangle_{W_2^5}\right|\\&\leq\|K_x^{(i)}(x)\|_{W_2^5}\|u(x)\|_{W_2^5}\\&\leq M_i\|u(x)\|_{W_2^5}\end{aligned}$$

Hence, for each $i=0,1,2,3,4$

$$\|u^{(i)}(x)\|_C\leq M_i\|u(x)\|_{W_2^5} \quad (12)$$

Theorem 3.3. The exact solution of Equations (5) and (6) is given by

$$u(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k,u(x_k),Tu(x_k))\bar{\psi}_i(x)$$

Proof. Let $u(x)$ be solution. Then, using Equation (11), we have

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty}\left\langle u(x),\overline{\psi}_i(x)\right\rangle_{W_2^5}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u(x),\psi_k(x)\right\rangle_{W_2^5}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u(x),L^{*}\varphi_k(x)\right\rangle_{W_2^5}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle Lu(x),\varphi_k(x)\right\rangle_{W_2^1}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle F\left(x,u(x),Tu(x)\right),\varphi_k(x)\right\rangle_{W_2^1}\overline{\psi}_i(x)
\end{aligned}$$

So, we have

$$u(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F\left(x_k,u(x_k),Tu(x_k)\right)\overline{\psi}_i(x) \tag{13}$$

To approximate the solution given in Equation (13), the first n-terms are taken

$$u_n(x)=\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}F\left(x_k,u(x_k),Tu(x_k)\right)\overline{\psi}_i(x) \tag{14}$$

Theorem 3.4. The approximated solution $u_n(x)$ converges uniformly to the exact solution $u(x)$. Also, for each $i=0,1,2,3,4$ $u_n^{(i)}(x)\to u^{(i)}(x)$ uniformly.

Proof. By Lemma 3.1, $\forall x\in[a,b]$

$$\begin{aligned}
\left|u_n(x)-u(x)\right| &= \left|\left\langle u_n(x)-u(x),K_x(x)\right\rangle_{W_2^5}\right|\\
&\le \left\|K_x(x)\right\|_{W_2^5}\left\|u_n(x)-u(x)\right\|_{W_2^5}\\
&\le M_0\left\|u_n(x)-u(x)\right\|_{W_2^5}
\end{aligned}$$

For the derivatives,

$$\begin{aligned}
\left|u_n^{(i)}(x)-u^{(i)}(x)\right| &= \left|\left\langle u_n(x)-u(x),K_x^{(i)}(x)\right\rangle_{W_2^5}\right|\\
&\le \left\|K_x^{(i)}(x)\right\|_{W_2^5}\left\|u_n(x)-u(x)\right\|_{W_2^5}\\
&\le M_i\left\|u_n(x)-u(x)\right\|_{W_2^5},\ M_i>0,\ i=1,2,3,4
\end{aligned}$$

Hence, if $\left\|u_n(x)-u(x)\right\|_{W_2^5}\to 0$ as $n\to\infty$, then $u_n(x)\to u(x)$ and $u_n^{(i)}(x)\to u^{(i)}(x)$ uniformly, $i=1,2,3,4$.

4. Representation of Numerical Solution

In this section, we are giving the numerical solutions form of Equations (5) and (6) in the space $W_2^5[0,1]$. Then, an iterative formulas for approximate solution is presented.

So as to inherit the behaviour of the numerical solution in the presented RKHS method the following results are guarantee this important requirements.

Lemma 4.1. If $u_n(x)\to u(x)$ in the sense of the norm of $W_2^5[0,1]$, $x_n\to y$ as $n\to\infty$, then as $n\to\infty$, we have

$$F\left(x_n,u_{n-1}(x_n),Tu_{n-1}(x_n)\right)\to F\left(y,u(y),Tu(y)\right)$$

Proof. In the first, we will show that $u_{n-1}\left(x_n\right)\to u\left(y\right)$ in the sense of the norm defined on the space $W_2^5\left[0,1\right]$. Since

$$\left|u_{n-1}\left(x_n\right)-u\left(y\right)\right|=\left|u_{n-1}\left(x_n\right)-u_{n-1}\left(y\right)+u_{n-1}\left(y\right)-u\left(y\right)\right|$$

By triangle inequality and the reproducing property of $K_x\left(y\right)$, we get

$$\begin{aligned}\left|u_{n-1}\left(x_n\right)-u_{n-1}\left(y\right)\right|&=\left|\left\langle u_{n-1}\left(x\right),K_{x_n}\left(x\right)-K_y\left(x\right)\right\rangle_{W_2^5}\right|\\&\le\left\|u_{n-1}\left(x\right)\right\|_{W_2^5}\left\|K_{x_n}\left(x\right)-K_y\left(x\right)\right\|_{W_2^5}\end{aligned}$$

So, we conclude that $\left\|K_{x_n}\left(x\right)-K_y\left(x\right)\right\|_{W_2^5}\to 0$ as $n\to\infty$.

Hence, $\left|u_{n-1}\left(x_n\right)-u_{n-1}\left(y\right)\right|\to 0$ as $x_n\to y$. Thus, for any $y\in\left[0,1\right]$, it holds that $\left|u_{n-1}\left(y\right)-u\left(y\right)\right|\to 0$ as $n\to\infty$. Therefore, $u_{n-1}\left(x_n\right)\to u\left(y\right)$ $n\to\infty$.

Thus, the continuity of J_1, J_2, and f, it is obtained that

$$\begin{aligned}&J_1\left(u_{n-1}\left(x_n\right)\right)\to J_1\left(u\left(y\right)\right),\\&J_2\left(u_{n-1}\left(x_n\right)\right)\to J_2\left(u\left(y\right)\right),\\&f\left(x_n,u_{n-1}\left(x_n\right)\right)\to f\left(y,u\left(y\right)\right)\end{aligned}$$

This shows that $Tu_{n-1}\left(x_n\right)\to Tu\left(y\right)$ as $n\to\infty$. Hence, we get the resuls.

The following theorem can be proved similarly.

Theorem 4.1. In the space $W_2^5\left[0,1\right]$ where $u_n\left(x\right)$ and $u\left(x\right)$ given in Equation (13) and Equation (14) we have

1) $u_n\left(x\right)\to u\left(x\right)$. Since $\left\{\psi_i\right\}$ is complete in $W_2^5\left[0,1\right]$, the partial sums u_n converge to u in the norm of $W_2^5\left[0,1\right]$.

2) Let $\varepsilon_n^2=\left\|u\left(x\right)-u_n\left(x\right)\right\|_{W_2^5}^2$ for each natural number n. Then, the sequence of numbers $\left\{\varepsilon_n\right\}$ is monotonic decreasing and converges to zero in the sense of the norm of $W_2^5\left[0,1\right]$.

Numerical Results

We are giving the complete numerical process tend to emphasize the implementation of algorithms.

Algorithm 4.1. To homogenized the constriants boundary condtions in Equation (3), we do the following:

Step 1: define $v\left(x\right)$ as

$$v\left(x\right)=u\left(x\right)-\phi\left(x\right)$$

where $\phi\left(x\right)$ is a function satisfying the conditions

$$\phi\left(0\right)=\alpha_1,\ \ \phi_1\left(1\right)=\beta_1,\ \ \phi'\left(0\right)=\alpha_2\ \text{ and }\ \phi'\left(1\right)=\beta_2.$$

Step 2: After easy calculations, one can obtain

$$\phi\left(x\right)=\left(\beta_1-\alpha_1-\alpha_2-\frac{3\beta_1-\beta_2}{2\alpha_2+3\alpha_1}\right)x^3+\frac{3\beta_1-\beta_2}{2\alpha_2+3\alpha_1}x^2+\alpha_2x+\alpha_1$$

Step 3: Substitute the new presented function $v\left(x\right)$ into Equations (1), (2) and boundary conditions (3) the problem transformed to the following IDE with homogenous BC's.

$$p_4(x)v^{(4)}+p_3(x)\left(v^{(3)}+\phi^{(3)}\right)(x)+p_2(x)\left(v^{(2)}+\phi^{(2)}\right)(x)+p_1(x)\left(\left(v^{(1)}+\phi^{(1)}\right)(x)\right)$$
$$=f\left(x,(v+\phi)(x)\right)+T\left[(v+\phi)\right](x)$$

subject to the homogeneous boundary conditions

$$v(0)=v(1)=0,$$
$$v''(0)=v''(1)=0.$$

Step 4: Write known function $v(x)$ as

$$u(x)=v(x)+\phi(x)$$

We mention here that this transformation will be applying on the numerical solution as wel as the exact solution.

To solve numerically Equations (1) and (2) subject to the boundary conditions (3) using the RKHS method we implement the following procedure.

Algorithm 4.2. To approximate the solution $u_n(t)$ of $u(t)$, we do the following:

Step a: Choose n assembling points in $[0,1]$;

Step b: Set $\psi_i(t)=L_sR_t(s)\big|_{s=t_i}$;

Step c: Find the coefficients β_{ik};

Step d: Set $\bar{\psi}_i(x)=\sum_{k=1}^{i}\beta_{ik}\psi_k(x)$, for $i=1,2,\cdots,n$;

Step e: Set $i=1$;

Step f: Set $u_i(x)=\sum_{j=1}^{i}\sum_{k=1}^{j}\beta_{ik}F\left(x_k,u(x_k),Tu(x_k)\right)\bar{\psi}_j(x)$;

Step g: If $i<n$, then set $i=i+1$ and go to step 6, else stop.

Example 1. Consider the linear equation:

$$x^4u^{(4)}(x)-xu'''(x)+u''(x)-\sin(x)u'(x)+u(x)=f(x)+T[u](x)$$

where

$$T[u](x)=\int_0^1(xt-1)u(t)\mathrm{d}t+\int_0^x\cosh(t)x^3u(t)\mathrm{d}t$$

subject to the boundary conditions

$$u(0)=u''(0)=1,$$
$$u(1)=u''(1)=\cosh(1),$$

where $0\le t<x\le 1$ and $f(x)$ is chosen such that the exact solution is $u(x)=\cosh(x)$

The approximate solution obtained using the RKM4 method is shown in Figure 1, while its derivative is illustrated in Figure 2 and the corresponding numerical solution is tabulated in Table 1.

Example 2. Consider the nonlinear equation:

$$x(x-1)u^{(4)}(x)-xu'''(x)-u(x)\mathrm{e}^{u(x)}=f(x)+T[u](x)$$

where

$$T[u](x)=\int_0^1 \mathrm{e}^x (t-1)\sinh(u(x))\,\mathrm{d}t+\int_0^x \cosh(x)t^3\mathrm{e}^{u(t)}\,\mathrm{d}t$$

subject to the boundary conditions

$$u(0)=0, u''(0)=1,$$

$$u(1)=\ln(2), u''(1)=-\frac{1}{4},$$

where $0\le t<x\le 1$ and $f(x)$ is chosen such that the exact solution is $u(x)=\ln(x+1)$

The approximate solution obtained using the RKM4 method is shown in Figure 3, while its derivative is illustrated in Figure 4 and the corresponding numerical solution is tabulated in Table 2.

Example 3. Consider the following nonlinear equation:

$$\sinh(x)u^{(4)}(x)-xu''(x)-(x-1)u'(x)+u^3(x)=f(x)+T[u](x),$$

in which the mixed operator is given as

$$T[u](x)=\int_0^1 (x-1)\ln(u(t))\,\mathrm{d}t+\int_0^x (x+1)^2 tu^3(t)\,\mathrm{d}t$$

and subject to the boundary conditions

$$u(0)=1, u(1)=\mathrm{e},$$

$$u'(0)=1, u'(1)=\mathrm{e},$$

where $0\le t<x\le 1$ and $f(x)$ is chosen such that the exact solution is

$$u(x)=\mathrm{e}^{-x}.$$

The approximate solution obtained using the RKM4 method is shown in Figure 5, while its derivative is illustrated in Figure 6 and the corresponding numerical solution is tabulated in Table 3.

Using all the prvious Algorithms, taking $x_i=\frac{i-1}{n-1}$, $i=1,2,\cdots,n$. The approximate solutions $u_n(x)$ of $u(x)$ at some specified points for $n=101$ are computed and tabulated.

Table 1. Numerical outcomes for Example1.

x	Exact solution	Approximate solution	Absolute error
0	1	1	0
0.1	1.00500417	1.005004305	1.35377662×10^{-7}
0.2	1.02006676	1.020074585	7.82499771×10^{-6}
0.3	1.04533851	1.045348268	9.75802745×10^{-6}
0.4	1.08107237	1.081079438	7.06834965×10^{-6}
0.5	1.12762597	1.127635882	9.91155212×10^{-6}
0.6	1.18546522	1.185474920	9.69967018×10^{-6}
0.7	1.25516901	1.255175274	6.26447249×10^{-6}

Continued

0.8	1.33743495	1.337438958	$4.00825596 \times 10^{-6}$
0.9	1.43308639	1.433089170	$2.77962568 \times 10^{-6}$
1	1.54308063	1.543080630	0

Table 2. Numerical outcomes for Example 2.

x	Exact solution	Approximate solution	Absolute error
0	0	0	0
0.1	0.09531018	0.095323680	$1.34997372 \times 10^{-5}$
0.2	0.18232156	0.182337252	$1.56924138 \times 10^{-5}$
0.3	0.26236426	0.262480352	$1.16091509 \times 10^{-4}$
0.4	0.33647224	0.336572943	$1.00703178 \times 10^{-4}$
0.5	0.40546511	0.405647270	$1.82160060 \times 10^{-4}$
0.6	0.47000363	0.470080576	$7.69462790 \times 10^{-5}$
0.7	0.53062825	0.531134512	$5.06262154 \times 10^{-4}$
0.8	0.58778666	0.587902612	$1.15951598 \times 10^{-4}$
0.9	0.64185389	0.642102879	$2.48989019 \times 10^{-4}$
1	0.69314718	0.693147180	0

Table 3. Numerical results for Example 3.

x	Exact solution	Approximate solution	Absolute error
0	1	1	0
0.1	0.90483742	0.904842776	$5.35609058 \times 10^{-6}$
0.2	0.81873075	0.818733318	$2.56787841 \times 10^{-6}$
0.3	0.74081822	0.740819437	$1.21697472 \times 10^{-6}$
0.4	0.67032005	0.670354405	$3.43549428 \times 10^{-5}$
0.5	0.60653066	0.606548251	$1.75913076 \times 10^{-5}$
0.6	0.54881164	0.548875657	$6.40170534 \times 10^{-5}$
0.7	0.49658530	0.496598855	$1.35553766 \times 10^{-5}$
0.8	0.44932896	0.449382572	$5.36120945 \times 10^{-5}$
0.9	0.40656966	0.406636500	$6.68397647 \times 10^{-5}$
1	0.36787944	0.367879440	0

Next, the geometric behaviours of the apptoximate solutions and the first derivative of the apptoximate solutions using the RKHS method are described for all listed problems, respectively.

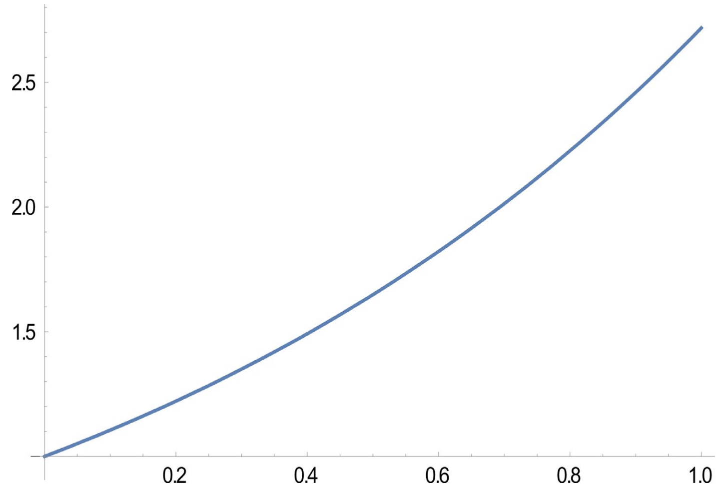

Figure 1. Skecth of the apptoximate solution using the RKHS method in Example 1.

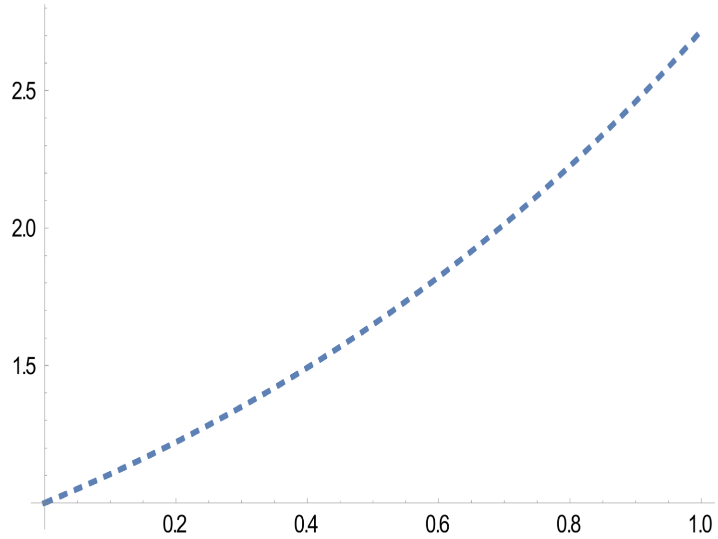

Figure 2. Skecth of the first derivative of the apptoximate solution using the RKHS method in Example 1.

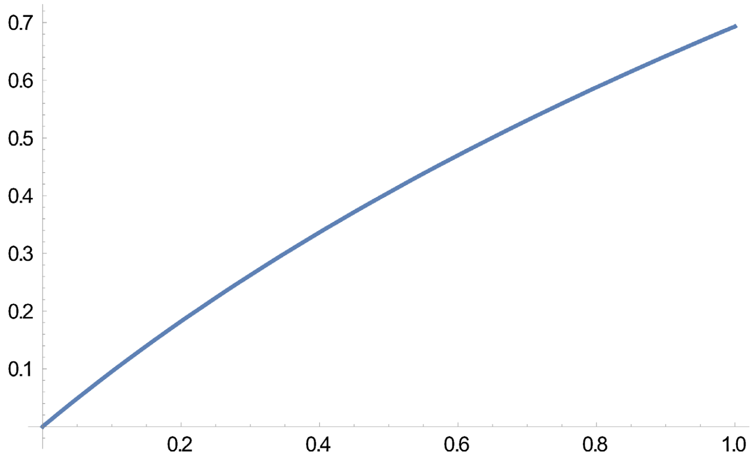

Figure 3. Skecth of the apptoximate solution using the RKHS method in Example 2.

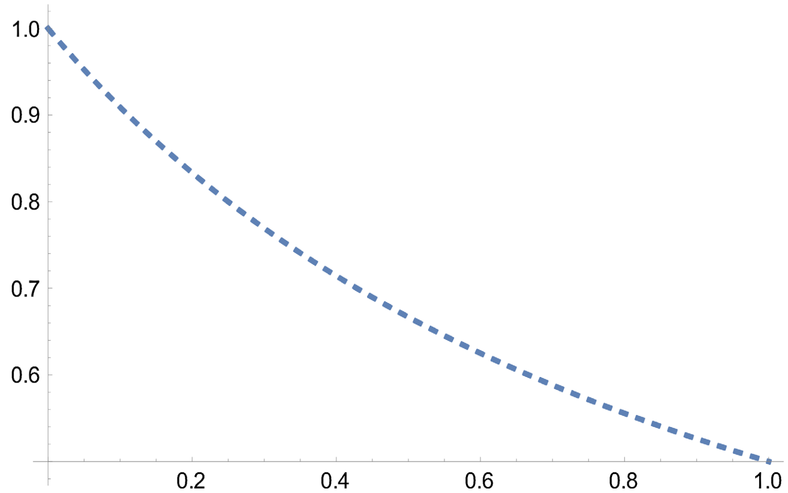

Figure 4. Skecth of the first derivative of the apptoximate solution using the RKHS method in Example 2.

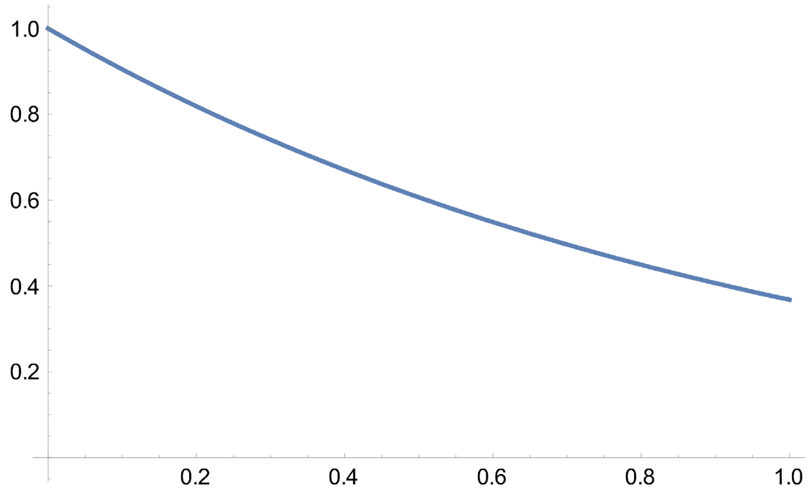

Figure 5. Skecth of the apptoximate solution using the RKHS method in Example 3.

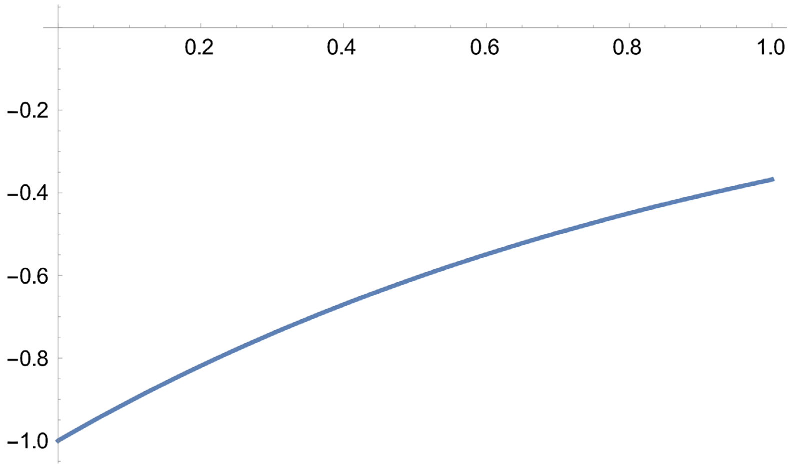

Figure 6. Skecth of the first derivative of the apptoximate solution using the RKHS method in Example 3..

5. Conclusions

This work was proposed and applied the RKHS method to solve three benchmark singular boundary value problems. The solution's methodology is grounded in generating orthonormal basis functions derived from the reproducing kernels. This orthonormal basis is constructed to facilitate the formulation and employ numerical solutions. Also, this constructed basis provides efficiently convergent approximate solutions. Additionally, both the approximate solution and its derivatives uniformly approach the exact solution and its respective derivatives.

The numerical results demonstrate the efficiency, reliability, and validity of the RKHS method, highlighting its strength and ease of handling these test cases. Meanwhile, from the given data, one can conclude that the validity of any numerical method depends on the complexity of the problem discussed.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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