

A Nonlinear Microdilation Microcontinuum Theory with Nonlinear and Linear Microconstituent Kinematics for Thermoelastic Solids

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Abstract

This paper presents the conservation and balance laws for nonlinear microdilation microcontinuum theory for thermoelastic solid medium in which microconstituents can only have pure volumetric deformation without distortion of shape. We consider nonlinear elasticity for microconstituents, for the medium as well as for interaction of the microconstituents with the solid medium. The theory is based on classical rotations as rigid body rotations of the microconstituents and use of balance of moment of moments balance law for thermodynamic equilibrium of the deforming matter. A check on the closure of the mathematical model consisting of conservation and the balance laws and the constitutive theories reveals that additional four equations are needed for closure of the mathematical model. It is shown that one of the four equations can be extracted from balance of angular momenta. We present two alternatives for the remaining three equations. In the first case, one could use Eringen's conservation of microinertia conservation law to obtain three equations, hence we have closure of the mathematical model. In the second alternative if we only consider linear volumetric deformation of the microconstituents, then we will only require one additional equation, hence the mathematical model has closure with additional one equation extracted from the balance of angular momenta. Pros and cons of both approaches are discussed in this paper from the point of view of thermodynamic and mathematical consistency of the resulting theory. Since the microconstituents are deformable, we begin derivation of the conservation and balance laws for the microconstituents followed by integral-average definitions that facilitate the derivation of macro conservation balance laws incorporating microconstituent kinematics. Constitutive theories are initiated using conjugate pairs in the entropy inequality

in conjunction with axiom of causality and are derived using representation theorem, hence ensuring their thermodynamic and mathematical consistency. Since the classical rotations and the conjugate moments is a new kinematically conjugate pair in the theory, the balance of moment of moments balance law is necessitated by classical thermodynamics for thermodynamic equilibrium of the microdilation solid medium. In the derivation of the conservation and the balance laws for the microdilation theory, we ensure that the modification of the conservation and the balance laws of classical continuum mechanics are supported by classical thermodynamics. The thermodynamically and mathematically consistent microdilation theory presented here is compared with Eringen's microstretch theory.

Keywords

Microstretch, Microdilation, Micro, Macro, Microcontinuum, Representation Theorem, New Balance Law, Conservation and Balance Laws, Integral-Average Definitions, Constitutive Theories, Thermodynamic, Mathematic, Consistency, Classical Thermodynamics

1. Introduction and Scope of Work

In a recent paper Surana *et al.* [1] has presented a thorough review of the published works on microcontinuum nonclassical theories. The work of references [2] [3], Eringen and Eringen *et al.* [4]-[21], references [22]-[29], Surana *et al.* [30]-[53] and references [54] [55] has been discussed in reference [1] and is not repeated here for the sake of brevity. From the literature review on microcontinuum theories it is rather clear that the published microcontinuum theories of Eringen consisting of conservation of balance laws and the constitutive theories are viewed to be valid microcontinuum theories. Extensive work published by Surana *et al.* on nonclassical continuum theories points out many inconsistencies, shortcomings, omissions, use of invalid measures and concepts in the published works that have lead to concerns regarding the validity of the microcontinuum theories published by Eringen and those following Eringen's work. Lack of thermodynamic and mathematical consistency of the published theories are of major concerns that have been reported in the works of Surana *et al.*

First, we discuss the details of the work presented in this paper. We begin by considering nonlinear kinematics of the microconstituents. This of course requires Green's strain measure $\mathcal{E}_{[0]}^{(\alpha)} = \frac{1}{2} \left(\left(\mathbf{J}^{(\alpha)} \right)^T \left(\mathbf{J}^{(\alpha)} \right) + \mathbf{I} \right)$. Thus, this nonlinear deformation measure necessitates that we consider $\mathbf{J}^{(\alpha)}$ for the microconstituents. If the microconstituents would have been completely deformable (as in micromorphic theory), then all nine components of the micro deformation gradient tensor $\mathbf{J}^{(\alpha)}$ would be deformational degrees of freedom for the microconstituents. However, in microdilation theory the microconstituents can only have pure vol-

umetric deformation without distortion of the shape of the microconstituents, thus this physics can be described by only four deformational degrees of freedom, volumetric strain or a quantity proportional to volumetric strain and three classical rotation ${}_c\Theta^{(\alpha)}$ in the volume $\bar{V}^{(\alpha)} + \partial\bar{V}^{(\alpha)}$ of the microconstituents.

$$\mathbf{J}^{(\alpha)} = \frac{1}{2} \left({}^d\mathbf{J}^{(\alpha)} + \left({}^d\mathbf{J}^{(\alpha)} \right)^T \right) + \frac{1}{2} \left({}^d\mathbf{J}^{(\alpha)} - \left({}^d\mathbf{J}^{(\alpha)} \right)^T \right) + \mathbf{I}$$

Special form of $\frac{1}{2} \left({}^d\mathbf{J}^{(\alpha)} + \left({}^d\mathbf{J}^{(\alpha)} \right)^T \right)$ defining pure volumetric deformation

is diagonal and contains only one degree of freedom, while $\frac{1}{2} \left({}^d\mathbf{J}^{(\alpha)} - \left({}^d\mathbf{J}^{(\alpha)} \right)^T \right)$ contains three classical rotation ${}_c\Theta^{(\alpha)}$ of the microconstituents. Thus a total of four deformational degrees of freedom completely define nonlinear microdilation physics of microconstituents.

A check on the closure of the mathematical model consisting of conservation and the balance laws and the constitutive theories reveals that additional four equations are needed for the closure of the mathematical model. In this paper, we show that one of these equations can be extracted from the balance of angular momenta. We still need three more equations for closure of the mathematical model. We consider two alternative approaches to address the issue of three additional equations: 1) In the first approach, we can use conservation of microinertia advocated by Eringen to obtain additional three equations. Now we have closure of the mathematical model in which the microconstituent have nonlinear microdilation deformation physics. The main problem with this approach is that conservation of microinertia conservation law is not supported by classical thermodynamics (there is no such conservation law in classical thermodynamics), thus this nonlinear microdilation theory is thermodynamically inconsistent. In view of the fact that all published microcontinuum theories (except those by the first author) are thermodynamically inconsistent and use conservation of microinertia conservation law whenever additional three equations are needed. This nonlinear microdilation theory may be of some value to those that are already using conservation of microinertia conservation law. Our view is that thermodynamic inconsistency of this microdilation theory suggests that this is not a valid microcontinuum theory, hence we do not advocate the use of this theory. 2) In the second approach, if we only consider linear kinematics of the microconstituents, then only one additional deformational degree of freedom (volumetric strain of the microconstituents) is required to define microconstituent deformation, hence only one additional equation is needed for closure which we have extracted from the balance of angular momenta. Thus, the mathematical model has closure. This microcontinuum theory is thermodynamically and mathematically consistent. The constitutive theories in the paper are presented for this microdilation theory in which microconstituent kinematics is linear. The classical rotations ${}_c\Theta^{(\alpha)}$ in the volume $\bar{V}^{(\alpha)} + \partial\bar{V}^{(\alpha)}$ remain as a free field, hence have no influence on the deformation physics of the microconstituents.

Since the microconstituents are deformable, the derivation of conservation and balance laws is initiated for the microconstituents, followed by integral-average definitions that facilitate derivation of macro conservation and balance laws incorporating micro deformation physics. The initial determination of the constitutive tensors and their argument tensors is made using conjugate pairs in the entropy inequality and the axiom of causality. The constitutive tensors and their argument tensors may be modified and/or augmented as desired to accommodate the physics that may not have been considered in the derivation of the entropy inequality. The constitutive theories are derived using representation theorem and integrity, hence are always complete and are thermodynamically and mathematically consistent. Material coefficients are derived in all cases followed by simplified forms of the constitutive theories. These are also compared with those of Eringen's.

The section following the constitutive theories highlights significant aspects of the work presented in this paper, various approaches used in the derivations that maintain mathematical and thermodynamic consistency of the microcontinuum theory presented here. In the subsequent section a brief discussion of Eringen's microstretch theory and various reasons for its thermodynamic and mathematical inconsistency are presented and discussed. This microdilation theory is not the same as Eringen's microstretch theory in which the microconstituent have stretching degree of freedom. Our view is that since in the microcontinuum theories the microconstituents are not oriented objects, hence the direction of stretch is not known. Thus, raising concerns about the theory. Some discussion is presented regarding the microstretch theory in section 9. Summary and conclusions are presented in the last section of the paper.

2. Micro Deformation, Deformable Material Point

In all microcontinuum theories micro deformation of the microconstituent influence macro response of the deforming solid. This is accomplished by assuming material point to be deformable, deformation of the material point is due to deformation of the microconstituents. Even though the concept of deformable material point is at odds with classical continuum mechanics, it is rationalized and implemented in rather simple way.

Consider the volume of matter subdivided into material points. Each material point containing microconstituents. A material point has mass and its center of mass is located in the fixed x -frame by $\mathbf{x}$ coordinates. We assume that the center of mass of the material point only sees the statistically average response of the microconstituents in its volume at the center of mass. We further assume that let these be a surrogate configuration of the microconstituents in the material volume in which each microconstituent has some response which in term is same as the statistically average response of the original configuration of microconstituent. With this assumption we need to consider only one microconstituent for micro deformation.

Figure 1 shows volume $V + \partial V$ and $\bar{V} + \partial\bar{V}$ of a material point in reference and deformed configuration with center of mass located at P and $\bar{P}$. The figure also shows microconstituent α with volume $V^{(\alpha)} + \partial V^{(\alpha)}$ and $\bar{V}^{(\alpha)} + \partial\bar{V}^{(\alpha)}$ in the undeformed and deformed configuration of microconstituent volume. Center of mass of the microconstituent is located at $\mathbf{x}^{(\alpha)}$ and $\bar{\mathbf{x}}^{(\alpha)}$ with respect to the x -frame and the center of mass of the material point in the reference configuration. $\bar{\mathbf{x}}^{(\alpha)}$ and $\bar{\mathbf{x}}^{(\alpha)}$ are of course corresponding location of the center of mass of the microconstituent in the current configuration. $\bar{\mathbf{x}}^{(\alpha)}$ is referred to as director in the undeformed configuration of a material point. $\bar{\mathbf{x}}^{(\alpha)}$ is its deformed director in the current configuration of the of the material point. Deformation of the director $\bar{\mathbf{x}}^{(\alpha)}$ is a measure of the deformation of the microconstituent α . Surana *et al.* [1] have derived nonlinear deformation measure using directors $\bar{\mathbf{x}}^{(\alpha)}$ and $\bar{\mathbf{x}}^{(\alpha)}$. Since in this theory there is only one director due to only one surrogate microconstituent, the microcontinuum theories derived using this are all called microcontinuum theories of grade 1. Microcontinuum theories of higher grade consider more than one director. Such theories are too complex to be of real value in applications. This derivation for a material point is valid for every material point when the matter is isotropic homogeneous.

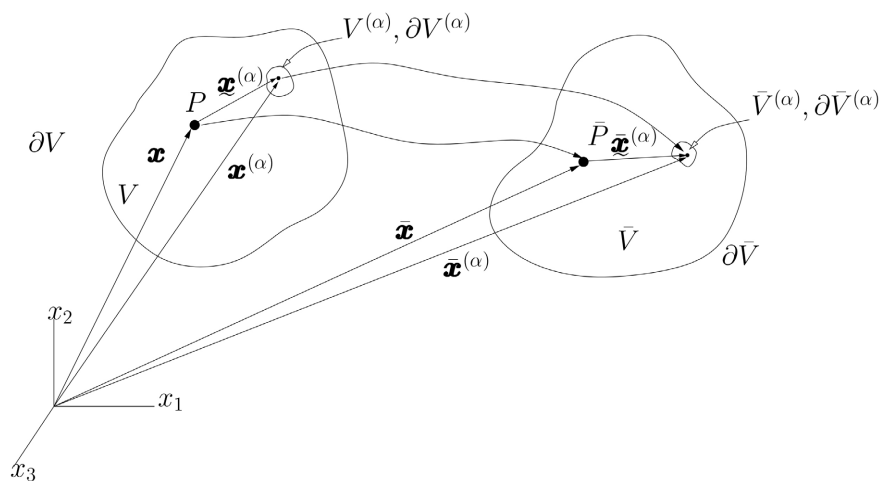


Figure 1. Undeformed and deformed configurations of material point volume.

3. Stress Tensor $\bar{S}$ Due to Micro Cauchy Stress Tensor $\bar{\sigma}^{(\alpha)}$

In the derivation of the conservation and the balance laws, we use the following integral-average definition.

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{mk}^{(\alpha)} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \bar{S}_{mk} d\bar{V} \quad (1)$$

In which $\bar{\sigma}^{(\alpha)}$ is the total Cauchy stress tensor for the microconstituent α , thus $\bar{S}_{mk}$ is total stress tensor. In this process there is no concept of additive decomposition of $\bar{\sigma}^{(\alpha)}$ into equilibrium and deviatoric stress tensors, hence volumetric and distortional physics are not considered explicitly. Secondly, microcon-

stituent density is eliminated through integral-average definitions. But, $\rho^{(\alpha)}$ that is needed if we were to consider constitutive theory for equilibrium stress for the microconstituents is not available anymore. Both of these consideration help us in concluding that the stress tensor $\bar{\mathbf{S}}$ or $\mathbf{S}$ is due to mechanical loading, hence is a function of work conjugate strain tensor and elastic properties of the microconstituents. Henceforth, we do not consider any additive decomposition of $\mathbf{S}$ but consider work conjugate strain tensor and temperature as its argument tensors of $\mathbf{S}$ in deriving the constitutive theory for it.

For the benefit of the readers not familiar with integral-average concept we present some material here that may be helpful. In integral-average definition the integral of quantity over the volume or area of the surface of the microconstituent defines the quantity to be used for the volume or the surface of the medium containing the microconstituent. This eliminates microconstituents and brings the integrated effects of the microconstituent kinematics in the macro physics.

4. Micro and Macro Stress and Moment Tensors

Additionally, in case of finite deformation, finite strain deformation physics, we need appropriate measures of stresses and strains. First, let us consider a microconstituent with $(V^{(\alpha)} + \partial V^{(\alpha)})$ and $(\bar{V}^{(\alpha)} + \partial \bar{V}^{(\alpha)})$ as undeformed and deformed volumes. Consider a tetrahedron $T^{(\alpha)}$ in undeformed configuration such that its oblique plane is part of $\partial V^{(\alpha)}$ and its other three orthogonal planes are parallel to the planes of the fixed x -frame. Upon finite deformation, finite strain, tetrahedron $T^{(\alpha)}$ deforms into $\bar{T}^{(\alpha)}$. The oblique plane of $\bar{T}^{(\alpha)}$ and its orientation changes compared to $T^{(\alpha)}$, and the edges of $\bar{T}^{(\alpha)}$ become curvilinear. If we assume that the tangent vectors to the curvilinear edges, covariant base vectors $\tilde{\mathbf{g}}_i^{(\alpha)}$ approximate the edges of the deformed tetrahedron, then its edges are now straight and the faces are flat but not orthogonal to each other and are not parallel to the planes of the fixed x -frame. If we choose Green's strain $\boldsymbol{\varepsilon}_{[0]}^{(\alpha)}$ as the finite strain measure for the microconstituent, a covariant measure using $\mathbf{J}^{(\alpha)}$ whose columns are covariant base vectors, then we must use faces of the deformed tetrahedron to define contravariant Cauchy stress tensor $\boldsymbol{\sigma}^{(\alpha)}$. The lower case brackets imply that it is Cauchy stress tensor, α means a typical microconstituent. In the following $*$ implies that it is the first Piola-Kirchhoff stress tensor. Zero means derivation of order zero *i.e.* the tensor itself. This notation is necessary to accommodate derivatives of the stress tensor of higher order needed for rheology. We define first and second contravariant Piola-Kirchhoff stress tensor ($\boldsymbol{\sigma}^{(\alpha)*}$ or $\boldsymbol{\sigma}^{[\alpha,0]}$) acting on $T^{(\alpha)}$ using $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ or $\boldsymbol{\sigma}^{(\alpha)}$. Following reference [35] [36], we can write the following.

$$\bar{\mathbf{P}}^{(\alpha)} = (\bar{\boldsymbol{\sigma}}^{(\alpha)})^T \cdot \bar{\mathbf{n}}^{(\alpha)} \quad (2)$$

Using correspondence rules:

$$\{d\bar{F}^{(\alpha)}\} = \{dF^{(\alpha)}\} \quad \text{for } \boldsymbol{\sigma}^{(\alpha)*} \quad (3)$$

$$\{d\bar{F}^{(\alpha)}\} = [J^{(\alpha)}] \{dF^{(\alpha)}\} \quad \text{for } \sigma^{[\alpha,0]} \quad (4)$$

We can write

$$[\sigma^{(\alpha)*}]^T = |J^{(\alpha)}| [\sigma^{(\alpha)}]^T [J^{(\alpha)}]^{-1} \quad (5)$$

$$[\sigma^{[\alpha,0]}]^T = |J^{(\alpha)}| [J^{(\alpha)}]^{-1} [\sigma^{(\alpha)}]^T [J^{(\alpha)}]^{-1} \quad (6)$$

and

$$[\sigma^{(\alpha)*}] = [\sigma^{(\alpha)}] [J^{(\alpha)}]^T \quad (7)$$

In deriving $\sigma^{(\alpha)*}$ the corresponding rule is $\{d\bar{F}\} = \{dF\}$, it implies that

$$(\sigma^{(\alpha)})^T \cdot n^{(\alpha)} = (\sigma^{(\alpha)*})^T \cdot n^{(\alpha)} \quad (8)$$

The stress and moment tensors $\sigma^{(0)}, \sigma^{[0]}, \sigma^*, m^*, m^{(0)}$ and $m^{[0]}$ are used as measures in macro deformation for which we use the following. Using corresponding rules [56] [57]

$$[\sigma^*]^T = |J| [\sigma^{(0)}]^T [J]^T]^{-1}; \quad [m^*]^T = |J| [m^{(0)}]^T [J]^T]^{-1} \quad (9)$$

$$[\sigma^{[0]}]^T = |J| [J]^{-1} [\sigma^{(0)}]^T [J]^T]^{-1}; \quad [m^{[0]}]^T = |J| [J]^{-1} [m^{(0)}]^T [J]^T]^{-1} \quad (10)$$

and

$$[\sigma^*]^T = [J] [\sigma^{[0]}]^T; \quad [m^*]^T = [J] [m^{[0]}]^T \quad (11)$$

We note that $\sigma^{(0)}$ is not symmetric, σ^* and $\sigma^{[0]}$ are nonsymmetric as well. When balance of moment of moments is used as a balance law $m^{(0)}$ is symmetric, hence $m^{(0)}$ is symmetric but m^* remains not symmetric.

5. Degrees of Freedom in Micro Deformation Physics in Nonlinear Microdilation Microcontinuum Theory

Consideration of the nonlinear deformation of microconstituents require that we consider second Piola-Kirchhoff stress tensor and rate of Green's strain tensor $\dot{\epsilon}_{[0]}^{(\alpha)}$ as rate of work conjugate pair. Consideration of $\epsilon_{[0]}^{(\alpha)}$ implies that we consider

$$\epsilon_{[0]}^{(\alpha)} = \frac{1}{2} \left([J^{(\alpha)}]^T [J^{(\alpha)}] - [I] \right) \quad (12)$$

and

$$J^{(\alpha)} = {}^d J^{(\alpha)} + I = {}^d J^{(\alpha)} + {}^d J^{(\alpha)} + I \quad (13)$$

Since the microconstituent deformation is purely volumetric ${}^d J^{(\alpha)}$ is a diagonal tensor in which each diagonal term is $\epsilon_v = \frac{1}{3}$ (volumetric strain). Thus, for this special purely volumetric deformation of the microconstituents we have

$$\mathbf{J}^{(\alpha)} = \varepsilon_v \boldsymbol{\delta} + {}^d_a \mathbf{J}^{(\alpha)} + \mathbf{I} \quad (14)$$

Using (14) we can define $\boldsymbol{\varepsilon}_{[0]}^{(\alpha)}$ in (12). From (12) we conclude that a microconstituent undergoing nonlinear deformation must have $\mathbf{J}^{(\alpha)}$ defined by (14) and $\boldsymbol{\varepsilon}_{[0]}^{(\alpha)}$ by (12) in which $\mathbf{J}^{(\alpha)}$ has four degrees of freedom, ε_v and three classical rotation ${}_c\boldsymbol{\Theta}^{(\alpha)}$ in the volume $V^{(\alpha)} + \partial V^{(\alpha)}$ of the microconstituent (α) associated with Green's strain measure $\boldsymbol{\varepsilon}_{[0]}^{(\alpha)}$. In addition, the microconstituents also have three rigid body rotations defined by ${}_c\boldsymbol{\Theta}$ (known). Hence, a microconstituent has a total of seven degrees of freedom: ε_v , ${}_c\boldsymbol{\Theta}^{(\alpha)}$ and ${}_c\boldsymbol{\Theta}$ in which ${}_c\boldsymbol{\Theta}$ are known.

6. Conservation and the Balance Laws

We derive conservation of mass, balance of linear momenta, balance of angular momenta, balance of moment of moments, first and second laws of thermodynamics in Eulerian as well as Lagrangian description for nonlinear elastic microdilation solid continua for finite deformation, finite strain physics. We begin with the conservation and balance laws derivation for microconstituents using thermodynamic framework of classical continuum mechanics. This is followed by introduction of integral-average definitions that holds at macro level and are used to derive valid conservation and balance laws for macro physics. The conservation and balance laws of classical continuum mechanics are used in the micro deformation physics. Due to use of integral-average definitions at macro level, the conservation and balance laws of classical continuum mechanics get modified when used for macro physics. Introduction of new kinematically conjugate pair of rotations and moments in addition to displacements and forces (or stresses) requires additional balance law, balance of moment of moments at the macro level only [39] [49] [58]. Contravariant second Piola-Kirchhoff stress tensor and the contravariant second Piola-Kirchhoff moment tensor are appropriate measures of stresses and moments for finite deformation physics. Since the first and second Piola-Kirchhoff stress tensor and the moment tensor are related to each other, it is more convenient to derive conservation and balance laws using first Piola-Kirchhoff tensors. In the end, we make substitutions of first Piola-Kirchhoff tensors in terms of second Piola-Kirchhoff tensors. Green's strain tensor is the correct strain measure for finite deformation, finite strain physics, hence is used in the present work.

6.1. Conservation of Mass

First we consider conservation of mass for microconstituent. For the microconstituent in the reference and the deformed configurations, conservation of mass can be expressed as:

$$\int_{V^{(\alpha)}} \rho_0^{(\alpha)} dV^{(\alpha)} = \int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \quad (15)$$

If microconstituent mass is conserved, then

$$\frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} = 0 \quad (16)$$

Using transport theorem [56] [57], we can write the following for (16).

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\frac{D}{Dt} \bar{\rho}^{(\alpha)}(\bar{\mathbf{x}}^{(\alpha)}, t) + \bar{\rho}^{(\alpha)}(\bar{\mathbf{x}}^{(\alpha)}, t) \frac{\partial \bar{V}_l^{(\alpha)}(\bar{\mathbf{x}}^{(\alpha)}, t)}{\partial \bar{x}_l^{(\alpha)}} \right) d\bar{V}^{(\alpha)} = 0 \quad (17)$$

Using localization theorem, we obtain the following from (17).

$$\frac{D}{Dt} \bar{\rho}^{(\alpha)} + \bar{\rho}^{(\alpha)} \frac{\partial \bar{V}_l^{(\alpha)}}{\partial \bar{x}_l^{(\alpha)}} = 0 \quad (18)$$

Equation (18) is the differential form of the continuity equation in Eulerian description for the microconstituent based on classical continuum mechanics.

In Lagrangian description, using (15) we can write

$$\int_{V^{(\alpha)}} \rho_0^{(\alpha)} dV^{(\alpha)} = \int_{V^{(\alpha)}} \rho^{(\alpha)} |\mathbf{J}^{(\alpha)}| dV^{(\alpha)} \quad (19)$$

Equation (19) implies that

$$\rho_0^{(\alpha)} = |\mathbf{J}^{(\alpha)}| \rho^{(\alpha)} \quad (20)$$

Equation (20) is continuity equation for microconstituent in Lagrangian description based on classical continuum mechanics.

Next, we consider conservation of mass at macro level. Consider Eulerian description in (15) and integration over $\bar{V}$ to obtain

$$\int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \quad (21)$$

Define

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \bar{\rho} d\bar{V} \quad (22)$$

Substituting (22) in (21) and setting its material derivative to zero (as mass is conserved for the volume $\bar{V}$).

$$\frac{D}{Dt} \int_{\bar{V}(t)} \bar{\rho} d\bar{V} = 0 \quad (23)$$

Using transport theorem [56] [57], we obtain the following from (23).

$$\int_{\bar{V}(t)} \left(\frac{D\bar{\rho}(\bar{\mathbf{x}}, t)}{Dt} + \bar{\rho}(\bar{\mathbf{x}}, t) \bar{\nabla} \cdot \bar{\mathbf{v}}(\bar{\mathbf{x}}, t) \right) d\bar{V} = 0 \quad (24)$$

Using localization theorem in (24)

$$\frac{D\bar{\rho}}{Dt} + \bar{\rho} \bar{\nabla} \cdot \bar{\mathbf{v}} = 0 \quad (25)$$

Equation (25) is the macro continuity equation in Eulerian description.

In Lagrangian Description we can write (23) as follows,

$$\int_{\bar{V}(t)} \bar{\rho} d\bar{V} = \int_V \rho_0 dV \quad (26)$$

Or

$$\int_V \rho |J| dV = \int_V \rho_0 dV \quad (27)$$

Using localization theorem we can obtain the following from (27).

$$\rho_0 = |J| \rho(\mathbf{x}, t) \quad (28)$$

Equation (28) is continuity equation resulting from the conservation of mass at macro level.

6.2. Balance of Linear Momenta

Consider balance of linear momenta for microconstituent. Based on classical continuum mechanics, we can write balance of linear momenta for a microconstituent in Eulerian description as follows, using $\bar{\sigma}^{(\alpha)}$ as microconstituent Cauchy stress in microdilation theory,

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - {}^b \bar{F}_k^{(\alpha)} \bar{\rho}^{(\alpha)} \right) d\bar{V}^{(\alpha)} - \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} = 0 \quad (29)$$

Using

$$\bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} = \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \quad (30)$$

We can write (29) in Lagrangian description as follows.

$$\int_{V^{(\alpha)}} \left(\rho_0^{(\alpha)} a_k^{(\alpha)} - {}^b F_k^{(\alpha)} \rho_0^{(\alpha)} \right) dV^{(\alpha)} - \int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} = 0 \quad (31)$$

Or

$$\int_{V^{(\alpha)}} \left(\rho_0^{(\alpha)} a_k^{(\alpha)} - {}^b F_k^{(\alpha)} \rho_0^{(\alpha)} - \sigma_{lk,l}^{(\alpha)*} \right) dV^{(\alpha)} = 0 \quad (32)$$

Using localization theorem [57]

$$\rho_0^{(\alpha)} a_k^{(\alpha)} - {}^b F_k^{(\alpha)} \rho_0^{(\alpha)} - \sigma_{lk,l}^{(\alpha)*} = 0 \quad (33)$$

Equation (33) is balance of linear momenta for microconstituent in the Lagrangian description in first Piola-Kirchhoff stress tensor.

Next, we consider derivation of macro balance of linear momenta.

Define

$$\int_{V^{(\alpha)}} \rho^{(\alpha)} a_k^{(\alpha)} dV^{(\alpha)} \stackrel{\text{def}}{=} \rho_0 a_k dV \quad (34)$$

$$\int_{V^{(\alpha)}} \left({}^b F_k^{(\alpha)} \right) \rho^{(\alpha)} dV^{(\alpha)} \stackrel{\text{def}}{=} \left({}^b F_k \right) \rho_0 dV \quad (35)$$

$$\int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} = \sigma_{lk}^* n_l dA \quad (36)$$

Using (34)-(36) in (31) and integrating over V and ∂V

$$\int_V \left(\rho_0 a_k - \rho_0 \left({}^b F_k \right) \right) dV - \int_{\partial V} \sigma_{lk}^* n_l dA = 0 \quad (37)$$

or

$$\int_V \left(\rho_0 a_k - \rho_0 \left({}^b F_k \right) - \sigma_{lk,l}^* \right) dV = 0 \quad (38)$$

Using localization theorem we obtained the following from (38)

$$\rho_0 a_k - \rho_0 \left({}^b F_k \right) - \sigma_{lk,l}^* = 0 \quad (39)$$

Equation (39) is balance of macro linear momenta in Lagrangian description in terms of first Piola-Kirchhoff stress tensor.

6.3. Balance of Macro Angular Momenta

We note from the following that there are three different possible forms that can be used to derive balance of macro angular momenta. We refer to the three as BAM1, BAM2, and BAM3. Conceptually and mathematically all three forms are the same *i.e.* can be obtained from each other, but there are some differences. The simplest way to derive this balance law is to consider balance of linear momenta, a statement of force balance and take its cross product with position or distance vector, integrate over the microconstituent volume and then integrate over the material point volume. Since cross product result in $\epsilon_{mkn} \bar{x}_m^{(\alpha)}$ we can multiply the balance of micro linear momenta by $\epsilon_{mkn} \bar{x}_m^{(\alpha)}$ instead of cross product. To this we must include the $\bar{M}^{(\alpha)}$ acting on $d\bar{A}^{(\alpha)}$. We give the three different form of balance of angular momenta in the following (BAM1, BAM2, and BAM3),

BAM1:

$$\begin{aligned} & \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \\ & - \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} - \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{M}_n^{(\alpha)} d\bar{A}^{(\alpha)} = 0 \end{aligned} \quad (40)$$

BAM2:

In this form the second term is expressed as volume integral over $\bar{V}^{(\alpha)}(t)$ and $\bar{V}(t)$

$$\begin{aligned} & \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \\ & - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} - \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{M}_n^{(\alpha)} d\bar{A}^{(\alpha)} = 0 \end{aligned} \quad (41)$$

BAM3:

To derive this form, we consider the following identity.

$$\left(\bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \right)_{,l} = \bar{x}_{m,l}^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} + \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} \quad (42)$$

$$\therefore \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} = \left(\bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \right)_{,l} - \bar{x}_{m,l}^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \quad (43)$$

We substitute from (43) in the second term of (41) to obtain the following

$$\begin{aligned}
& \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \\
& - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\epsilon_{mkn} \left(\bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \right)_{,l} - \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \right) d\bar{V}^{(\alpha)} \\
& - \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{M}_n^{(\alpha)} d\bar{A}^{(\alpha)} = 0
\end{aligned} \quad (44)$$

Equation (44) is the third possible form that can be used to derive macro balance of angular momenta. The motivation for using identity (43) in the balance of angular momenta of the microconstituents is discussed in section 6.8.2

6.3.1. Balance of Angular Momenta Using BAM1

In this case we consider (40).

Consider the first term in (40) (say T1) *i.e.*

$$T1 = \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \quad (45)$$

Let

$$\bar{x}_m^{(\alpha)} = \bar{x}_m + \bar{x}_m^{(\alpha)} \quad (46)$$

Substitute from (46) in (45)

$$\begin{aligned}
T1 &= \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \\
&+ \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)}
\end{aligned} \quad (47)$$

$$\begin{aligned}
T1 &= \int_{\bar{V}(t)} \epsilon_{mkn} \bar{x}_m \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \\
&+ \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)}
\end{aligned} \quad (48)$$

Define

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \left(\bar{\rho} \bar{a}_k - \bar{\rho} \left({}^b \bar{F}_k \right) \right) d\bar{V} \quad (49)$$

Using (49) in (48), we can write

$$\begin{aligned}
T1 &= \int_{\bar{V}(t)} \epsilon_{mkn} \bar{x}_m \left(\bar{\rho} \bar{a}_k - \bar{\rho} \left({}^b \bar{F}_k \right) \right) d\bar{V} \\
&+ \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \left(\bar{\rho}^{(\alpha)} \bar{a}_k^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_k^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)}
\end{aligned} \quad (50)$$

In Lagrangian description (50) can be written as

$$\begin{aligned}
T1 &= \int_V \epsilon_{mkn} x_m \left(\rho_0 a_k - \rho_0 \left({}^b F_k \right) \right) dV \\
&+ \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} x_m^{(\alpha)} \left(\rho_0^{(\alpha)} a_k^{(\alpha)} - \rho_0^{(\alpha)} \left({}^b F_k^{(\alpha)} \right) \right) dV^{(\alpha)}
\end{aligned} \quad (51)$$

Consider second term in (40) (say T2)

$$T2 = \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \quad (52)$$

Substituting $\bar{x}_m^{(\alpha)}$ from (46) in (52)

$$T2 = \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \left(\bar{x}_m + \bar{x}_m^{(\alpha)} \right) \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \quad (53)$$

$$\begin{aligned} T2 &= \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \\ &= \int_{\partial \bar{V}(t)} \epsilon_{mkn} \bar{x}_m \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \end{aligned} \quad (54)$$

Note that

$$\bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} = \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \quad (55)$$

Using (55) in (54)

$$T2 = \int_{\partial V} \epsilon_{mkn} x_m \int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} + \int_{\partial V} \int_{\partial V^{(\alpha)}} \epsilon_{mkn} x_m^{(\alpha)} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \quad (56)$$

Define

$$\int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \stackrel{\text{def}}{=} \sigma_{lk}^* n_l dA \quad (57)$$

Using (57) in (56)

$$\begin{aligned} T2 &= \int_{\partial V} \epsilon_{mkn} x_m \sigma_{lk}^* n_l dA + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} \left(x_m^{(\alpha)} \sigma_{lk}^{(\alpha)*} \right)_{,l} dV^{(\alpha)} \\ &= \int_V \epsilon_{mkn} \left(x_m \sigma_{lk}^* \right)_{,l} dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} \left(x_{m,l}^{(\alpha)} \sigma_{lk}^{(\alpha)*} + x_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} \right) dV^{(\alpha)} \\ &= \int_V \epsilon_{mkn} \left(x_{m,l} \sigma_{lk}^* + x_m \sigma_{lk,l}^* \right) dV + \int_V \epsilon_{mkn} \int_{V^{(\alpha)}} \left(\sigma_{mk}^{(\alpha)*} + x_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} \right) dV^{(\alpha)} \\ &= \int_V \left(\epsilon_{mkn} \sigma_{mk}^* + x_m \sigma_{lk,l}^* \right) dV + \int_V \epsilon_{mkn} \int_{V^{(\alpha)}} \left(\sigma_{mk}^{(\alpha)*} + x_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} \right) dV^{(\alpha)} \end{aligned} \quad (58)$$

Define

$$\int_{V^{(\alpha)}} \sigma_{mk}^{(\alpha)*} dV^{(\alpha)} \stackrel{\text{def}}{=} S_{mk}^* dV \quad (59)$$

Using (59) in (58)

$$T2 = \int_V \epsilon_{mkn} \left(\sigma_{mk}^* + x_m \sigma_{lk,l}^* + S_{mk}^* \right) dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} x_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} dV^{(\alpha)} \quad (60)$$

This is the final form of the second term in (40) (T2) for BAM1.

Lastly, consider third term in (40) (say T3) *i.e.*

$$T3 = \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{M}_n^{(\alpha)} d\bar{A}^{(\alpha)} \quad (61)$$

$$T3 = \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{m}_{ln}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \quad (62)$$

Define

$$\therefore \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{m}_l^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} = \int_{\partial V^{(\alpha)}} m_{ln}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \stackrel{\text{def}}{=} m_{ln}^* n_l dA \quad (63)$$

Using (63) in (62) we can write (62) as follows

$$T3 = \int_{\partial V} m_{ln}^* n_l dA = \int_V m_{ln,l}^* dV \quad (64)$$

Substituting T1, T2 and T3 from (51), (60), and (64) in (40) for the three terms, we can write the following for balance of angular momenta (BAM1)

$$\begin{aligned} & \int_V \epsilon_{mkn} x_m \left(\rho_0 a_k - \rho_0 \left({}^b F_k \right) \right) dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} x_m^{(\alpha)} \left(\rho_0^{(\alpha)} a_k^{(\alpha)} - \rho_0^{(\alpha)} \left({}^b F_k^{(\alpha)} \right) \right) dV^{(\alpha)} \\ & - \int_V \epsilon_{mkn} \left(\sigma_{mk}^* + x_m \sigma_{lk,l}^* + S_{mk}^* \right) dV - \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} x_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} dV^{(\alpha)} - \int_V m_{ln,l}^* dV = 0 \end{aligned} \quad (65)$$

Grouping terms in

$$\begin{aligned} & \int_V \epsilon_{mkn} x_m \left(\rho_0 a_k - \rho_0 \left({}^b F_k \right) - \sigma_{lk,l}^* \right) dV \\ & + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} x_m^{(\alpha)} \left(\rho_0^{(\alpha)} a_k^{(\alpha)} - \rho_0^{(\alpha)} \left({}^b F_k^{(\alpha)} \right) - \sigma_{lk,l}^{(\alpha)*} \right) dV^{(\alpha)} \\ & - \int_V \left(\epsilon_{mkn} \left(\sigma_{mk}^* + S_{mk}^* \right) + m_{ln,l}^* \right) dV = 0 \end{aligned} \quad (66)$$

The first and second terms in (66) are zero due to macro and micro balance of linear momenta, thus (66) reduces to

$$\int_V \left(\epsilon_{mkn} \left(\sigma_{mk}^* + S_{mk}^* \right) + m_{ln,l}^* \right) dV = 0 \quad (67)$$

Using localization theorem (67) yields

$$\epsilon_{mkn} \left(\sigma_{mk}^* + S_{mk}^* \right) + m_{ln,l}^* = 0 \quad (68)$$

Equation (68) is the final form of balance of macro angular momenta for BAM1.

6.3.2. Balance of Angular Momenta BAM2

In this case we consider (41). Since the first and the third terms in (41) are same as the first and the third term in (40), Equation (51) and (64) holds for the first and third terms of (41). We only need to consider the second term of (41) (say t2)

$$t2 = \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} \quad (69)$$

Substituting $\bar{x}_m^{(\alpha)}$ from (46) in (69)

$$\begin{aligned} t2 &= \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \left(\bar{x}_m + \bar{x}_m^{(\alpha)} \right) \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} \\ &= \int_{\bar{V}(t)} \epsilon_{mkn} \bar{x}_m \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} \\ &= \int_{\partial \bar{V}(t)} \epsilon_{mkn} \bar{x}_m \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\bar{V}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk,l}^{(\alpha)} d\bar{V}^{(\alpha)} \end{aligned} \quad (70)$$

Define

$$\int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \stackrel{\text{def}}{=} \int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \stackrel{\text{def}}{=} \sigma_{lk}^* n_l dA \quad (71)$$

Using (71) in (70)

$$\begin{aligned} t_2 &= \int_{\partial V} \epsilon_{mkn} x_m \sigma_{lk}^* n_l dA + \int_{\partial \bar{V}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \\ &= \int_V \epsilon_{mkn} (x_m \sigma_{lk}^*)_{,l} dV + \int_{\partial V} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \\ &= \int_V \epsilon_{mkn} (x_{m,l} \sigma_{lk}^* + x_m \sigma_{lk,l}^*) dV + \int_{\partial V} \epsilon_{mkn} \int_{\partial V^{(\alpha)}} \bar{x}_m^{(\alpha)} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \\ &= \int_V \epsilon_{mkn} (\sigma_{mk}^* + x_m \sigma_{lk,l}^*) dV + \int_V \epsilon_{mkn} \int_{V^{(\alpha)}} (\bar{x}_m^{(\alpha)} \sigma_{lk}^{(\alpha)*})_{,l} dV^{(\alpha)} \\ &= \int_V \epsilon_{mkn} (\sigma_{mk}^* + x_m \sigma_{lk,l}^*) dV + \int_V \epsilon_{mkn} \int_{V^{(\alpha)}} (\bar{x}_{m,l}^{(\alpha)} \sigma_{lk}^{(\alpha)*} + \bar{x}_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*}) dV^{(\alpha)} \\ &= \int_V \epsilon_{mkn} (\sigma_{mk}^* + x_m \sigma_{lk,l}^*) dV + \int_V \epsilon_{mkn} \int_{V^{(\alpha)}} \sigma_{mk}^{(\alpha)*} dV^{(\alpha)} + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} dV^{(\alpha)} \end{aligned} \quad (72)$$

Define

$$\int_{V^{(\alpha)}} \sigma_{mk}^{(\alpha)*} dV^{(\alpha)} \stackrel{\text{def}}{=} S_{mk}^* dV \quad (73)$$

Using (73) in (72)

$$t_2 = \int_V \epsilon_{mkn} (\sigma_{mk}^* + x_m \sigma_{lk,l}^* + S_{mk}^*) dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} dV^{(\alpha)} \quad (74)$$

As expected, t_2 for BAM2 in (74) is exactly same as in (60) for BAM1.

Substituting from (51) and (64) for the first and the last term in (41) and using (74) for the second term of (41), we obtain exactly the same expression for BAM2 as (65) for BAM1, then following derivation in BAM1 (Equation (65)-(68)) we obtain the same final expression for balance of angular momenta as (68) *i.e.*

$$\epsilon_{mkn} (\sigma_{mk}^* + S_{mk}^*) + m_{ln,l}^* = 0 \quad (75)$$

6.3.3. Balance of Angular Momenta Using BAM3

Consider Equation (44).

The first and the last term in (44) are also exactly the same as the first and last term in (40) (BAM1), hence Equation (51) and (64) hold for these two terms of (44), we consider the second term of (44) in the following

$$tt_2 = \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} (\bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)})_{,l} d\bar{V}^{(\alpha)} - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_{m,l}^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} d\bar{V}^{(\alpha)} \quad (76)$$

$$tt_2 = \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_m^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{x}_{m,l}^{(\alpha)} \bar{\sigma}_{lk}^{(\alpha)} d\bar{V}^{(\alpha)} \quad (77)$$

Substituting for $\bar{x}_m^{(\alpha)}$ from (46) in (77)

$$tt_2 = \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \epsilon_{mkn} (\bar{x}_m + \bar{x}_m^{(\alpha)}) \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \epsilon_{mkn} \bar{\sigma}_{mk}^{(\alpha)} d\bar{V}^{(\alpha)} \quad (78)$$

Let

$$\int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} = \int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \quad (79)$$

$$\int_{\bar{V}^{(\alpha)}(t)} \bar{\sigma}_{mk}^{(\alpha)} d\bar{V}^{(\alpha)} = \int_{V^{(\alpha)}} \sigma_{mk}^{(\alpha)*} dV^{(\alpha)} \quad (80)$$

Using (79) and (80) in (78)

$$\begin{aligned} \text{tt2} = & \int_{\partial V} \epsilon_{mkn} x_m \int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} + \int_{\partial V} \epsilon_{mkn} \int_{\partial V^{(\alpha)}} x_m^{(\alpha)} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \\ & - \int_V \epsilon_{mkn} \int_{V^{(\alpha)}} \sigma_{mk}^{(\alpha)*} dV^{(\alpha)} \end{aligned} \quad (81)$$

Define

$$\int_{V^{(\alpha)}} \sigma_{mk}^{(\alpha)*} dV^{(\alpha)} \stackrel{\text{def}}{=} S_{mk}^* dV \quad (82)$$

$$\int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \stackrel{\text{def}}{=} \sigma_{lk}^* n_l dA \quad (83)$$

Using (82) and (83) in (81)

$$\begin{aligned} \text{tt2} = & \int_{\partial V} \epsilon_{mkn} x_m \sigma_{lk}^* n_l dA + \int_{\partial V} \epsilon_{mkn} x_m^{(\alpha)} \int_{\partial V^{(\alpha)}} \sigma_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} - \int_V \epsilon_{mkn} S_{mk}^* dV \\ = & \int_V \epsilon_{mkn} (x_m \sigma_{lk}^*)_{,l} dV + \int_V \epsilon_{mkn} \int_{V^{(\alpha)}} x_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} dV^{(\alpha)} - \int_V \epsilon_{mkn} S_{mk}^* dV \\ = & \int_V \epsilon_{mkn} (x_{m,l} \sigma_{lk}^* + x_m \sigma_{lk,l}^*) dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} x_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} dV^{(\alpha)} - \int_V \epsilon_{mkn} S_{mk}^* dV \\ = & \int_V \epsilon_{mkn} (\sigma_{mk}^* + x_m \sigma_{lk,l}^*) dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} x_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} dV^{(\alpha)} - \int_V \epsilon_{mkn} S_{mk}^* dV \end{aligned} \quad (84)$$

Or

$$\text{tt2} = \int_V \epsilon_{mkn} (\sigma_{mk}^* + x_m \sigma_{lk,l}^* - S_{mk}^*) dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} x_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} dV^{(\alpha)} \quad (85)$$

Substituting from (51) and (64) for the first and the third terms in (44) and using (85) for the second term we can write the following.

$$\begin{aligned} & \int_V \epsilon_{mkn} x_m (\rho_0 a_k - \rho_0 ({}^b F_k)) dV + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} x_m^{(\alpha)} (\rho_0^{(\alpha)} a_k^{(\alpha)} - \rho_0^{(\alpha)} ({}^b F_k^{(\alpha)})) dV^{(\alpha)} \\ & - \int_V \epsilon_{mkn} (\sigma_{mk}^* + x_m \sigma_{lk,l}^* - S_{mk}^*) dV - \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} x_m^{(\alpha)} \sigma_{lk,l}^{(\alpha)*} dV^{(\alpha)} - \int_V m_{ln,l}^* dV = 0 \end{aligned} \quad (86)$$

Collecting terms in (86)

$$\begin{aligned} & \int_V \epsilon_{mkn} x_m (\rho_0 a_k - \rho_0 ({}^b F_k) - \sigma_{lk,l}^*) dV \\ & + \int_V \int_{V^{(\alpha)}} \epsilon_{mkn} x_m^{(\alpha)} (\rho_0^{(\alpha)} a_k^{(\alpha)} - \rho_0^{(\alpha)} ({}^b F_k^{(\alpha)}) - \sigma_{lk,l}^{(\alpha)*}) dV^{(\alpha)} \\ & - \int_V (\epsilon_{mkn} (\sigma_{mk}^* - S_{mk}^*) + m_{ln,l}^*) dV = 0 \end{aligned} \quad (87)$$

The first two terms in (87) are zero due to balance of macro and micro balance of

linear momenta. Thus, (87) reduces to

$$\int_V \left(\epsilon_{mkn} (\sigma_{mk}^* - S_{mk}^*) + m_{ln,l}^* \right) dV = 0 \quad (88)$$

Using localization theorem (88) yields

$$\epsilon_{mkn} (\sigma_{mk}^* - S_{mk}^*) + m_{ln,l}^* = 0 \quad (89)$$

This is the final form of balance of macro angular momenta for BAM3.

We make the following observations and remarks.

We note that BAM1 and BAM2 use actual balance of micro linear momenta in the derivation whereas in BAM3, the micro gradient stress term in the balance of micro linear momenta is altered using an identity. The result is the negative sign for S^* term in the balance of angular momenta (Equation (89)). Equation (89) is what is derived by Eringen using a weighting function $\phi^{(\alpha)}$ for the balance of micro linear momenta. The derivation presented here for BAM3 shows that weighting function is not needed as the end result of using the weighting function is same as what have presented. The answer to the question of whether the correct form of balance of angular momenta is (68) (or (75)) or (89) is important.

Based on the derivation presented in BAM1 and BAM2 without the weight function (or identity) that follows standard approach of deriving balance of angular momenta, there is little reason to doubt the outcome. In BAM3 when identity is used to substitute for one of the term in balance of angular momenta, the result is negative sign for S^* . At this stage we maintain both positive and negative signs in balance of angular momenta and consider the following (in Lagrangian description)

$$\epsilon_{mkn} (\sigma_{mk}^* \pm S_{mk}^*) + m_{ln,l}^* = 0 \quad (90)$$

We defer the decision on which sign to choose for S_{mk}^* until we have finished the derivation of all balance laws. We recall that $S^{(0)}$ is a diagonal tensor with all diagonal being same, hence S^* and $S^{[0]} \cdot J^T (= S^*)$ are diagonal tensors. At this stage we do not need to convert σ^* and S^* into $\sigma^{[0]}$ and $S^{[0]}$. Since

$$S_{mk}^* \neq S_{km}^*, \quad \epsilon_{mkn} S_{mk}^* \neq 0 \quad (91)$$

Hence from (90) we can only conclude that

$$\epsilon_{mkn} (\sigma_{mk}^*) + m_{ln,l}^* \neq 0 \quad (92)$$

Thus in balance of angular momenta (92) holds in which σ^* and m^* need to be expressed in terms of $\sigma^{[0]}$ and $m^{[0]}$.

6.4. First Law of Thermodynamics

Since the conservation and the balance laws of classical continuum mechanics hold for micro deformation of the microconstituents, we can begin with the energy equation for the microconstituents over volume $\bar{V}^{(\alpha)}$ with boundary $\partial \bar{V}^{(\alpha)}$ and integrate over $\bar{V}$ and $\partial \bar{V}$,

$$\begin{aligned} & \int_{\vec{V}(t)} \int_{\vec{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} \dot{\bar{e}}^{(\alpha)} d\vec{V}^{(\alpha)} - \int_{\vec{V}(t)} \int_{\vec{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{v}_{l,k}^{(\alpha)} d\vec{V}^{(\alpha)} + \int_{\vec{V}(t)} \int_{\vec{V}^{(\alpha)}(t)} \bar{q}_{k,k}^{(\alpha)} d\vec{V}^{(\alpha)} \\ & - \int_{\partial \vec{V}(t)} \int_{\partial \vec{V}^{(\alpha)}(t)} \bar{M}_k^{(\alpha)} \left({}^r_c \bar{\Theta} \right)_k d\bar{A}^{(\alpha)} = 0 \end{aligned} \quad (93)$$

In which $\bar{e}^{(\alpha)}$ is the specific internal energy, $\bar{q}^{(\alpha)}$ is heat flux and ${}^r_c \bar{\Theta}$ are classical rotation rates (due to $\bar{\nabla}^{(\alpha)} \times \bar{v}^{(\alpha)}$). We consider each term in (93).

Consider first term in (93) (say t1)

$$t1 = \int_{\vec{V}(t)} \int_{\vec{V}^{(\alpha)}(t)} \bar{\rho}^{(\alpha)} \dot{\bar{e}}^{(\alpha)} d\vec{V}^{(\alpha)} = \int_V \int_{V^{(\alpha)}} \rho_0^{(\alpha)} \dot{e}^{(\alpha)} dV^{(\alpha)} = \int_V \frac{D}{Dt} \int_{V^{(\alpha)}} \rho_0^{(\alpha)} e^{(\alpha)} dV^{(\alpha)} \quad (94)$$

Let

$$\int_{V^{(\alpha)}} \rho_0^{(\alpha)} e^{(\alpha)} dV^{(\alpha)} \stackrel{\text{def}}{=} \rho_0 e dV \quad (95)$$

Substituting (95) in (94)

$$t1 = \int_V \frac{D}{Dt} (\rho_0 e) dV \stackrel{\text{def}}{=} \int_V \rho_0 \dot{e} dV \quad (96)$$

Consider second term of (93) ($\bar{\sigma}^{(\alpha)}$ is symmetric, $\bar{\sigma}_{lk}^{(\alpha)} = \bar{\sigma}_{kl}^{(\alpha)}$)

$$\begin{aligned} t2 &= \int_{\vec{V}(t)} \int_{\vec{V}^{(\alpha)}(t)} \bar{\sigma}_{lk}^{(\alpha)} \bar{v}_{l,k}^{(\alpha)} d\vec{V}^{(\alpha)} = \int_{\vec{V}(t)} \int_{\vec{V}^{(\alpha)}(t)} \left(\left(\bar{\sigma}_{lk}^{(\alpha)} \bar{v}_l^{(\alpha)} \right)_{,k} - \bar{\sigma}_{kl,k}^{(\alpha)} \bar{v}_l^{(\alpha)} \right) d\vec{V}^{(\alpha)} \\ &= \int_{\partial \vec{V}(t)} \int_{\partial \vec{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{v}_l^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} - \int_{\vec{V}(t)} \int_{\vec{V}^{(\alpha)}(t)} \bar{\sigma}_{kl,k}^{(\alpha)} \bar{v}_l^{(\alpha)} d\vec{V}^{(\alpha)} \end{aligned} \quad (97)$$

We note

$$\bar{v}_l^{(\alpha)} = \bar{v}_l + \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} \quad (98)$$

and

$$\bar{\sigma}_{kl,k}^{(\alpha)} = \bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_l^{(\alpha)} \right) \quad (99)$$

Substituting (98) and (99) in (97)

$$\begin{aligned} t2 &= \int_{\partial \vec{V}(t)} \int_{\partial \vec{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \left(\bar{v}_l + \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} \right) \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \\ &\quad - \int_{\vec{V}(t)} \int_{\vec{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_l^{(\alpha)} \right) \right) \left(\bar{v}_l + \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} \right) d\vec{V}^{(\alpha)} \end{aligned} \quad (100)$$

$$\begin{aligned} t2 &= \int_{\partial \vec{V}(t)} \bar{v}_l \int_{\partial \vec{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\partial \vec{V}(t)} \int_{\partial \vec{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \\ &\quad - \int_{\vec{V}(t)} \bar{v}_l \int_{\vec{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_l^{(\alpha)} \right) \right) d\vec{V}^{(\alpha)} \\ &\quad - \int_{\vec{V}(t)} \int_{\vec{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b \bar{F}_l^{(\alpha)} \right) \right) \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} d\vec{V}^{(\alpha)} \end{aligned} \quad (101)$$

Define

$$\int_{\partial \vec{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} = \int_{\partial V^{(\alpha)}} \sigma_{kl}^{(\alpha)*} n_k^{(\alpha)} dA^{(\alpha)} \stackrel{\text{def}}{=} \sigma_{kl}^* n_k dA \quad (102)$$

and

$$\int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b F_l^{(\alpha)} \right) \right) d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \left(\bar{\rho} \bar{a}_l - \bar{\rho} \left({}^b F_l \right) \right) d\bar{V} \quad (103)$$

Substituting (102) and (103) in (101)

$$\begin{aligned} t2 = & \int_{\partial V} \sigma_{kl}^* v_l n_k dA + \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{\sigma}_{kl}^{(\alpha)} \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \\ & - \int_V v_l \left(\rho_0 a_l - \rho_0 \left({}^b F_l \right) \right) dV \\ & - \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \left(\bar{\rho}^{(\alpha)} \bar{a}_l^{(\alpha)} - \bar{\rho}^{(\alpha)} \left({}^b F_l^{(\alpha)} \right) \right) \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} d\bar{V}^{(\alpha)} \end{aligned} \quad (104)$$

We note the following

$$\int_{\partial V} \sigma_{kl}^* v_l n_k dA = \int_V \left(\sigma_{kl}^* v_l \right)_{,k} dV = \int_V \left(v_l \sigma_{kl,k}^* + \sigma_{kl}^* v_{l,k} \right) dV \quad (105)$$

and

$$\begin{aligned} \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{L}_{lm}^{(\alpha)} \bar{x}_m^{(\alpha)} \bar{\sigma}_{kl}^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} &= \int_{\partial V} \int_{\partial V^{(\alpha)}} L_{lm}^{(\alpha)} x_m^{(\alpha)} \sigma_{kl}^{(\alpha)*} n_k^{(\alpha)} dA^{(\alpha)} \\ &= \int_{\partial V} L_{lm}^{(\alpha)} \int_{\partial V^{(\alpha)}} x_m^{(\alpha)} \sigma_{kl}^{(\alpha)*} n_k^{(\alpha)} dA^{(\alpha)} \\ &= \int_V \int_{V^{(\alpha)}} L_{lm}^{(\alpha)} \left(\sigma_{kl}^{(\alpha)*} x_m^{(\alpha)} \right)_{,k} dV^{(\alpha)} \\ &= \int_V \int_{V^{(\alpha)}} L_{lm}^{(\alpha)} \left(\left(\sigma_{kl,k}^{(\alpha)*} \right) x_m^{(\alpha)} + \sigma_{kl}^{(\alpha)*} x_{m,k}^{(\alpha)} \right) dV^{(\alpha)} \\ &= \int_V \int_{V^{(\alpha)}} L_{lm}^{(\alpha)} \left(\sigma_{kl,k}^{(\alpha)*} x_m^{(\alpha)} + \sigma_{ml}^{(\alpha)*} \right) dV^{(\alpha)} \end{aligned} \quad (106)$$

Define

$$\int_{V^{(\alpha)}} L_{lm}^{(\alpha)} \sigma_{ml}^{(\alpha)*} dV^{(\alpha)} \stackrel{\text{def}}{=} S_{ml}^* L_{lm}^{(\alpha)} dV \quad (107)$$

Substituting (105) in (104) and (107) in (106) and then (106) in (104) and converting last term in (104) to Lagrangian description.

$$\begin{aligned} t2 = & \int_V \left(v_l \sigma_{kl,k}^* + \sigma_{kl}^* v_{l,k} \right) dV + \int_V S_{ml}^* L_{lm}^{(\alpha)} dV + \int_V \int_{V^{(\alpha)}} \left(L_{lm}^{(\alpha)} \sigma_{kl,k}^{(\alpha)*} \right) x_m^{(\alpha)} dV^{(\alpha)} \\ & - \int_V v_l \left(\rho_0 a_l - \rho_0 \left({}^b F_l \right) \right) dV - \int_V \int_{V^{(\alpha)}} \left(\rho_0^{(\alpha)} a_l^{(\alpha)} - \rho_0^{(\alpha)} \left({}^b F_l^{(\alpha)} \right) \right) L_{lm}^{(\alpha)} x_m^{(\alpha)} dV^{(\alpha)} \end{aligned} \quad (108)$$

Collecting term

$$\begin{aligned} t2 = & - \int_V v_l \left(\rho_0 a_l - \rho_0 \left({}^b F_l \right) - \sigma_{kl,k}^* \right) dV \\ & - \int_V \int_{V^{(\alpha)}} \left(\rho_0^{(\alpha)} a_l^{(\alpha)} - \rho_0^{(\alpha)} \left({}^b F_l^{(\alpha)} \right) - \sigma_{kl,k}^{(\alpha)*} \right) L_{lm}^{(\alpha)} x_m^{(\alpha)} dV^{(\alpha)} \\ & + \int_V \left(S_{ml}^* L_{lm}^{(\alpha)} + \sigma_{kl}^* v_{l,k} \right) dV \end{aligned} \quad (109)$$

The first and the second term in (109) are zero due to balance of micro and macro

linear momenta, hence (109) reduces to the following

$$t_2 = \int_V \left(S_{ml}^* \dot{J}_{lm}^{(\alpha)} + \sigma_{kl}^* \dot{J}_{lk} \right) dV \quad (110)$$

Consider third term in (93) (say t_3)

$$t_3 = \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{q}_{k,k}^{(\alpha)} d\bar{V}^{(\alpha)} = \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{q}_k^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \quad (111)$$

Define

$$\int_{\partial \bar{V}^{(\alpha)}(t)} \bar{q}_k^{(\alpha)} \bar{n}_k^{(\alpha)} d\bar{A}^{(\alpha)} \stackrel{\text{def}}{=} \bar{q}_k \bar{n}_k d\bar{A} \quad (112)$$

Using (112) in (111)

$$t_3 = \int_{\partial \bar{V}(t)} \bar{q}_k \bar{n}_k d\bar{A} = \int_{\bar{V}(t)} \bar{q}_{k,k} d\bar{V} = \int_V q_{k,k} dV \quad (113)$$

Consider the fourth term in (93) (say t_4)

$$\begin{aligned} t_4 &= \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{M}_k^{(\alpha)} \left({}^r \bar{\Theta} \right)_k d\bar{A}^{(\alpha)} \\ &= \int_{\partial \bar{V}(t)} \left({}^r \bar{\Theta} \right)_k \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{m}_{lk}^{(\alpha)} \bar{n}_l^{(\alpha)} d\bar{A}^{(\alpha)} \\ &= \int_{\partial V} {}^c \dot{\Theta}_k \int_{\partial V^{(\alpha)}} m_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \end{aligned} \quad (114)$$

Let

$$\int_{\partial V^{(\alpha)}} m_{lk}^{(\alpha)*} n_l^{(\alpha)} dA^{(\alpha)} \stackrel{\text{def}}{=} m_{lk}^* n_l dA \quad (115)$$

Using (115) in (114)

$$\begin{aligned} t_4 &= \int_{\partial V} {}^c \dot{\Theta}_k m_{lk}^* n_l dA = \int_{\partial V} {}^c \dot{\Theta} \cdot \left(\mathbf{m}^* \right)^T \cdot dA \\ &= \int_V \nabla \cdot \left({}^c \dot{\Theta}_r \cdot \left(\mathbf{m}^* \right)^T \right) dV \end{aligned} \quad (116)$$

A simple calculation shows [30]-[52] [56] [57]

$$\nabla \cdot \left({}^c \dot{\Theta} \cdot \left(\mathbf{m}^* \right)^T \right) = {}^c \dot{\Theta} \cdot \left(\nabla \cdot \mathbf{m}^* \right) + \mathbf{m}^* : {}^c \Theta \dot{\mathbf{J}} \quad (117)$$

Using (117) in (116)

$$t_4 = \int_V \left({}^c \dot{\Theta} \cdot \left(\nabla \cdot \mathbf{m}^* \right) + \mathbf{m}^* : {}^c \Theta \dot{\mathbf{J}} \right) dV \quad (118)$$

Substituting t_1 , t_2 , t_3 , and t_4 from (96), (110), (113), and (118) we can write (93) as follows.

$$\begin{aligned} &\int_V \rho_0 \dot{e} dV - \int_V \left(S_{ml}^* \dot{J}_{lm}^{(\alpha)} + \sigma_{kl}^* \dot{J}_{lk} \right) dV + \int_V q_{k,k} dV \\ &- \int_V \left({}^c \dot{\Theta} \cdot \left(\nabla \cdot \mathbf{m}^* \right) + \mathbf{m}^* : {}^c \Theta \dot{\mathbf{J}} \right) dV = 0 \end{aligned} \quad (119)$$

Or

$$\int_V \left(\rho_0 \dot{e} - \mathbf{S}^* : \dot{\mathbf{J}}^{(\alpha)} - \boldsymbol{\sigma}^* : \dot{\mathbf{J}} + q_{k,k} - \left({}_c \dot{\boldsymbol{\Theta}} \cdot (\nabla \cdot \mathbf{m}^*) \right) + \mathbf{m}^* : {}^c \boldsymbol{\Theta} \dot{\mathbf{J}} \right) dV \quad (120)$$

Using localization theorem we have

$$\rho_0 \dot{e} - \mathbf{S}^* : \dot{\mathbf{J}}^{(\alpha)} - \boldsymbol{\sigma}^* : \dot{\mathbf{J}} + q_{k,k} - \left({}_c \dot{\boldsymbol{\Theta}} \cdot (\nabla \cdot \mathbf{m}^*) \right) + \mathbf{m}^* : {}^c \boldsymbol{\Theta} \dot{\mathbf{J}} = 0 \quad (121)$$

This is the macro energy equation in Lagrangian description.

6.5. Second Law of Thermodynamics

The rate of increase of entropy for a microconstituent volume $\bar{V}^{(\alpha)}$ due to entropy imparted by contacting or noncontacting sources is given by

$$\frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\eta}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \geq \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{h}^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\bar{V}^{(\alpha)}(t)} \bar{s}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \quad (122)$$

$\bar{\eta}^{(\alpha)}$ is the entropy density of the volume $\bar{V}^{(\alpha)}$ of the microconstituent, $\bar{h}^{(\alpha)}$ is the entropy flux imparted to volume $\bar{V}^{(\alpha)}$ through $\partial \bar{V}^{(\alpha)}$ by the surrounding medium through contact and $\bar{s}^{(\alpha)}$ is the source of entropy in $\bar{V}^{(\alpha)}$ due to non-contacting sources or bodies. Integrating (122) over $\bar{V}$

$$\int_{\bar{V}(t)} \frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\eta}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \geq \int_{\partial \bar{V}(t)} \int_{\partial \bar{V}^{(\alpha)}(t)} \bar{h}^{(\alpha)} d\bar{A}^{(\alpha)} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \bar{s}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \quad (123)$$

Using

$$\bar{h}^{(\alpha)} = -\bar{\Psi}_k^{(\alpha)} \bar{n}_k^{(\alpha)} \quad (124)$$

$$\bar{\Psi}_k^{(\alpha)} = \frac{\bar{q}_k^{(\alpha)}}{\bar{\theta}} \quad (125)$$

$$\therefore \bar{h}^{(\alpha)} = -\frac{\bar{q}_k^{(\alpha)} \bar{n}_k^{(\alpha)}}{\bar{\theta}} \quad (126)$$

and

$$\bar{s}^{(\alpha)} = \frac{\bar{r}^{(\alpha)}}{\bar{\theta}} \quad (127)$$

Using (126) and (127) in (123)

$$\begin{aligned} & \int_{\bar{V}(t)} \frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\eta}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \\ & \geq \int_{\partial \bar{V}(t)} - \int_{\partial \bar{V}^{(\alpha)}(t)} \frac{\bar{q}_k^{(\alpha)} \bar{n}_k^{(\alpha)}}{\bar{\theta}} d\bar{A}^{(\alpha)} + \int_{\bar{V}(t)} \int_{\bar{V}^{(\alpha)}(t)} \frac{\bar{r}^{(\alpha)} \bar{\rho}^{(\alpha)}}{\bar{\theta}} d\bar{V}^{(\alpha)} \end{aligned} \quad (128)$$

Define

$$\frac{D}{Dt} \int_{\bar{V}^{(\alpha)}(t)} \bar{\eta}^{(\alpha)} \bar{\rho}^{(\alpha)} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \dot{\bar{\eta}} \bar{\rho} d\bar{V} \quad (129)$$

$$\int_{\bar{V}^{(\alpha)}(t)} \frac{\bar{q}_k^{(\alpha)} \bar{n}_k^{(\alpha)}}{\bar{\theta}} d\bar{A}^{(\alpha)} \stackrel{\text{def}}{=} \frac{\bar{q}_k \bar{n}_k}{\bar{\theta}} d\bar{A} \quad (130)$$

and

$$\int_{\bar{V}^{(\alpha)}(t)} \frac{\bar{r}^{(\alpha)} \bar{\rho}^{(\alpha)}}{\bar{\theta}} d\bar{V}^{(\alpha)} \stackrel{\text{def}}{=} \frac{\bar{r} \bar{\rho}}{\bar{\theta}} d\bar{V} \quad (131)$$

Substituting from (129)-(131) in (128)

$$\int_{\bar{V}(t)} \bar{\rho} \dot{\bar{\eta}} d\bar{V} \geq - \int_{\partial \bar{V}(t)} \left(\frac{\bar{q}_k}{\bar{\theta}} \right) \bar{n}_k d\bar{A} + \int_{\bar{V}(t)} \frac{\bar{r} \bar{\rho}}{\bar{\theta}} d\bar{V} \quad (132)$$

Converting (132) to Lagrangian description

$$\int_V \rho_0 \dot{\eta} dV \geq - \int_{\partial V} \left(\frac{q_k}{\theta} \right) n_k dA + \int_V \frac{r \rho_0}{\theta} dV \quad (133)$$

or

$$\int_V \rho_0 \dot{\eta} dV \geq - \int_V \left(\frac{q_k}{\theta} \right)_k dV + \int_V \frac{r \rho_0}{\theta} dV \quad (134)$$

or

$$\int_V \left(\rho_0 \dot{\eta} + \frac{q_{k,k}}{\theta} - \frac{q_k \theta_{,k}}{\theta^2} + \frac{r \rho_0}{\theta} \right) dV \geq 0 \quad (135)$$

Using localization theorem

$$\rho_0 \dot{\eta} + \frac{q_{k,k}}{\theta} - \frac{q_k \theta_{,k}}{\theta^2} + \frac{r \rho_0}{\theta} \geq 0 \quad (136)$$

Multiply (136) throughout by θ

$$\rho_0 \dot{\eta} \theta + q_{k,k} - \frac{q_k \theta_{,k}}{\theta} + r \rho_0 \geq 0 \quad (137)$$

Let

$$\Phi = e - \eta \theta \quad (138)$$

$$\dot{\Phi} = \dot{e} - \dot{\eta} \theta - \eta \dot{\theta} \quad (139)$$

$$\therefore \rho_0 \theta \dot{\eta} = \rho_0 \dot{e} - \rho_0 \dot{\Phi} - \rho_0 \eta \dot{\theta} \quad (140)$$

Substituting from (140) into (137)

$$-\rho_0 (\dot{\Phi} + \eta \dot{\theta}) + \rho_0 \dot{e} + q_{k,k} - \frac{q_k \theta_{,k}}{\theta} - r \rho_0 \geq 0 \quad (141)$$

Substituting $\rho \dot{e}$ from energy Equation (121) after inserting radiation term $\rho_0 r$

$$-\rho_0 (\dot{\Phi} + \eta \dot{\theta}) + \left(\mathbf{S}^* : \dot{\mathbf{J}}^{(\alpha)} + \boldsymbol{\sigma}^* : \dot{\mathbf{J}} - q_{k,k} + \left(\left({}_c \dot{\Phi} \cdot (\nabla \cdot \mathbf{m}^*) \right) + \mathbf{m}^* : {}^c \dot{\mathbf{J}} \right) + \rho_0 r \right) + q_{k,k} - \frac{q_k \theta_{,k}}{\theta} - r \rho_0 \geq 0 \quad (142)$$

$q_{k,k}$ and $r \rho_0$ term cancel and we can write the following after changing the sign.

$$\rho_0 (\dot{\Phi} + \eta \dot{\theta}) - \boldsymbol{\sigma}^* : \dot{\mathbf{J}} - \mathbf{S}^* : \dot{\mathbf{J}}^{(\alpha)} - \frac{q_k \theta_{,k}}{\theta} - \left({}_c \dot{\Phi} \cdot (\nabla \cdot \mathbf{m}^*) + \mathbf{m}^* : {}^c \dot{\mathbf{J}} \right) \leq 0 \quad (143)$$

Inequality (143) is the entropy inequality in Lagrangian description resulting from the second law of thermodynamics.

6.6. Balance of Moment of Moments Balance Law

In classical continuum mechanics for solid medium based on classical thermodynamics the displacements and forces co-exists as a kinematically conjugate pair. Displacements are kinematic variables and forces are conjugate quantities to the kinematic variables. For a kinematically conjugate pair the classical thermodynamics requires two balance laws for the thermodynamic equilibrium of the deforming volume of matter. The first balance law is the balance of conjugate quantities *i.e.* forces and the second balance law is the moment of the conjugate quantities, that is balance of moment of the forces. These two balance law are of course balance of linear momenta and the balance of angular momenta. In the absence of either one of these balance laws the deforming volume of matter is not in thermodynamic equilibrium.

In the microcontinuum theories for solid matter in addition to displacement and forces as a kinematically conjugate pair we also have classical rotation ${}_{\mathcal{C}}\Theta$ and moment $\mathbf{M}$ as a second kinematically conjugate pair. Thus, based on classical thermodynamics each kinematically conjugate pair requires two balance laws: balance of conjugate quantities and balance of moment of the conjugate quantity. Thus, for the two kinematically conjugate pair we need: 1) balance of forces and balance of moment of forces due to displacement and forces as a kinematically conjugate pair, these are balance of linear and balance of angular momenta 2) balance of moments and balance of moment of moments due to classical rotation ${}_{\mathcal{C}}\Theta$ and moments $\mathbf{M}$ as a second kinematically conjugate pair. Balance of moments is the same as balance of angular momenta that already exists due to displacement and forces as a kinematically conjugate pair, hence can be modified to include nonclassical physics. This modification is obviously supported by classical thermodynamics. The second balance law, balance of moment of moments is a new balance law needed for thermodynamic equilibrium of the deforming microcontinuum volume of matter. This balance law was first proposed by Yang *et al.* [58] as a statement of static equilibrium. Surana *et al.* [30] [50] have presented derivation of this balance law based on rate considerations for solid and fluent media. They showed that the outcome of this balance law is that the Cauchy moment tensor is symmetric in the microcontinuum theories.

$$\epsilon_{ijk} \bar{m}_{ij} = 0 \quad \text{or} \quad \epsilon_{ijk} m_{ij} = 0 \quad (144)$$

Details of the derivation are omitted here but are given in references [30] [51]. We remark that in the absence of this balance law (case for almost all published works on microcontinuum theories):

1) The deforming solid microcontinuum is not in thermodynamic equilibrium as this balance law is a requirement based on classical thermodynamics.

2) The outcome of this balance law (144) is that Cauchy moment tensor is symmetric. This has serious consequences in the derivation of the constitutive theories for the moment tensor. When this balance law is used, the constitutive theory is required only for symmetric Cauchy moment tensor.

3) In the absence of this balance law, the Cauchy moment tensor is nonsymmetric. This results in spurious conjugate pairs in the entropy inequality that necessitate nonphysical and invalid constitutive theories as demonstrated by Surana *et al.* [30] [50].

In the absence of this balance law a thermodynamically and mathematically consistent and physically valid microcontinuum theory is not possible.

6.7. Summary of Macro Conservation and Balance Laws in Lagrangian Description

$$\rho_0(\mathbf{x}) = |\mathbf{J}| \rho(\mathbf{x}, t) \quad (145)$$

$$\rho_0 a_k - \rho_0 \left({}^b F_k \right) - \sigma_{lk,k}^* = 0 \quad (146)$$

$$\epsilon_{mkn} \left(\sigma_{mk}^* \pm S_{mk}^* \right) + m_{ln,l}^* = 0 \quad (147)$$

$$\rho_0 \dot{e} - \sigma^* : \dot{\mathbf{J}} - \mathbf{S}^* : \dot{\mathbf{J}}^{(\alpha)} + \nabla \cdot \mathbf{q} - \left({}_c \dot{\Theta} \cdot (\nabla \cdot \mathbf{m}^*) + \mathbf{m}^* : {}^c \Theta \dot{\mathbf{J}} \right) \leq 0 \quad (148)$$

$$\rho_0 \left(\dot{\Phi} + \eta \dot{\theta} \right) - \sigma^* : \dot{\mathbf{J}} - \mathbf{S}^* : \dot{\mathbf{J}}^{(\alpha)} + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - \left({}_c \dot{\Theta} \cdot (\nabla \cdot \mathbf{m}^*) + \mathbf{m}^* : {}^c \Theta \dot{\mathbf{J}} \right) \leq 0 \quad (149)$$

$$\epsilon_{ijk} m_{ij} = 0 \quad (150)$$

This mathematical model consists of seven partial differential equations: balance of linear momenta (3), balance of angular momenta (3), and energy equation (1) in twenty nine dependent variables: $\mathbf{u}(3)$, $\sigma(9)$, $\mathbf{S}(1)$, $\mathbf{m}(6)$, $\mathbf{q}(3)$, $\theta(1)$, $\mathbf{J}^{(\alpha)}(6)$ thus, additional seventeen equations are needed for closure. Constitutive theories provide sixteen equations: $\sigma(6)$, $\mathbf{S}(1)$, $\mathbf{m}(6)$, $\mathbf{q}(3)$. Thus, additional one equation is needed for closure. This is discussed in section 6.8.

6.8. Additional Equation in the Mathematical Model

Eringen [4]-[21] and those following his work suggest that in the derivation of the balance of angular momenta, the *permutation tensor must be dropped* to obtain another balance law, moments of ${}_s \mathbf{S}^*$ and ${}_s \sigma^*$ that must balance with gradients of the symmetric part of the moment tensor $\mathbf{m}^*$. In Eringen's work, the non-symmetric moment tensor also contains permutation tensor in balance of angular momenta hence yields three equations containing gradients of skew symmetric part of moment tensor and the skew symmetric part of stress tensors σ^* and $\mathbf{S}^*$ and the additional one equation is obtained by balancing moment of the symmetric parts of the traces of σ^* and $\mathbf{S}^*$ with the gradient of the symmetric part of third rank moment tensor. Eringen suggests that these four equations are sufficient to address four degrees of freedom for the microconstituents, three unknown rigid rotations ${}_a \Theta$, and a pure stretch. There are many concerns and issues in this approach. First, definition of moment tensor using $\sigma^{(\alpha)}$ (classical mechanics) is incorrect because moment tensor is conjugate to rigid rotations of the microconstituents hence has no relationship to $\sigma^{(\alpha)}$. Due to lack of use of balance of moment of moments balance laws, the moment tensor is nonsymmet-

ric and has incorrect definition. Permutation tensor also appears with the moment tensor. This is incorrect as permutation is a consequence of cross product of force vectors with distance vectors; moments do not require this cross product. Dropping permutation tensor is not a reasonable proposition when the permutation tensor should not appear with the moment tensor in the first place. Because of all these issues both balance of angular momenta and additional equations proposed by Eringen are not valid and are in error.

6.8.1. Derivation of Additional One Equation

In the following we first consider a more general case of microconstituent deformation than microdilation in which the microconstituents have six deformational degree of freedom (as in micromorphic theory [1]). In the end we specialize these results for our case of nonlinear microdilation theory. We begin with balance of angular momenta (147) and note that σ^* has nine independent components ${}_a\sigma^*$ and ${}_s\sigma^*$ and S^* is a symmetric tensor hence has nine independent component as well.

$$\epsilon_{mkn}(\sigma_{mk}^* \pm S_{mk}^*) + m_{ln,l}^* = 0 \quad (151)$$

The presence of permutation tensor in (151) forces us to discard symmetric components of σ^* and tensor S^* and we are only left with ${}_a\sigma^*$ balanced by the gradients of m^* , we note that the relationship between ${}_s\sigma^*$ and S^* is implicitly present in (151) but is eliminated due to presence of permutation tensor. We premultiply (151) by $(\epsilon_{mkn})^{-1}$, inverse of ϵ_{mkn} .

$$(\epsilon_{mkn})^{-1}(\epsilon_{mkn})(\sigma_{mk}^* \pm S_{mk}^*) + (\epsilon_{mkn})^{-1}m_{ln,l}^* = 0 \quad (152)$$

or

$$\sigma_{mk}^* \pm S_{mk}^* + (\epsilon_{mkn})^{-1}m_{ln,l}^* = 0 \quad (153)$$

But inverse of ϵ_{mkn} is ϵ_{mkn} (for values 1, 2, 3 for m, k, n). Thus we can write (153) as

$$\sigma_{mk}^* \pm S_{mk}^* + \epsilon_{mkn}m_{ln,l}^* = 0 \quad (154)$$

Consider additive decomposition of σ_{mk}^* into ${}_a\sigma_{mk}^*$ and ${}_s\sigma_{mk}^*$.

$${}_a\sigma_{mk}^* + {}_s\sigma_{mk}^* \pm {}_aS_{mk}^* + {}_sS_{mk}^* + \epsilon_{mkn}m_{ln,l}^* = 0 \quad (155)$$

Since

$${}_aS_{mk}^* + {}_a\sigma_{mk}^* + \epsilon_{mkn}m_{ln,l}^* = 0 \quad (156)$$

Equation (155) reduces to

$${}_s\sigma^* \pm {}_sS^* = 0 \quad (157)$$

At this stage choice of negative sign is physical as it would imply ${}_s\sigma^* = {}_sS^*$, thus we can write (157) as

$${}_s\sigma^* - {}_sS^* = 0 \quad (158)$$

Additional equation needed for closure is obtained using the following obtained

from (158)

$$\text{tr}({}_s\boldsymbol{\sigma}^*) - \text{tr}({}_s\mathbf{S}^*) = 0 \quad (159)$$

This provides additional equations necessary for the closure of the mathematical model.

Also we note that balance of angular momenta in classical continuum mechanics is a statement of balance of moment of the forces. Addition of moment tensor due to nonclassical mechanics to this balance law is justified without much explanation. This balance law works perfectly for micropolar case in which the microconstituents can only have rigid rotations that cause the moment tensor. When microconstituents are deformable volume average stress tensor $\mathbf{S}$ results from the Cauchy stress tensor $\bar{\boldsymbol{\sigma}}^{(\alpha)}$ or $\boldsymbol{\sigma}^{(\alpha)}$ of the microconstituents. At the interface between the microconstituent and the medium ${}_s\mathbf{S}^* = {}_s\boldsymbol{\sigma}^*$ must hold (static equilibrium), but this physics is not present in the derivation of the balance angular momenta. $\mathbf{S}^*$ is treated as another stress tensor like $\boldsymbol{\sigma}^*$ (as in BAM1 and BAM2) hence will naturally have positive sign in the balance of angular momenta in BAM1 and BAM2. Use of identity in BAM3 changes the sign from positive to negative for the term that is used to obtain volume average $\mathbf{S}$, thus we are able to obtain the desired equation with negative sign for ${}_s\mathbf{S}^*$ that is supported by the physics at the interface between the microconstituents and the medium. Balance of angular momenta remain as in (151) with the positive sign for $\mathbf{S}^*$

6.8.2. Further Explanation of Using Identity for BAM3 in Balance of Angular Momenta and The Negative Sign for $\mathbf{S}^*$ in (159)

In Section 6.3 the third form of balance of angular momenta (BAM3) makes use of identity (43). The consequence of this is that in the resulting balance of angular momenta $\mathbf{S}^*$ has negative sign. Even though the rationale for (159) given above is perfectly valid, but a stronger support in favor of (159) is perhaps more desirable.

In classical continuum mechanics the balance of angular momenta (consider infinitesimal theory for illustration purposes) $\epsilon_{ijk}\bar{\sigma}_{ij} = \epsilon_{ijk}\sigma_{ij} = 0$ is simply a statement that establishes symmetry of the Cauchy stress tensor. When this balance law is considered for microcontinuum theories its modifications require addition of ${}_a\boldsymbol{\sigma}$ and $\mathbf{m}$ in the balance law, both of which are due to microcontinuum physics and their addition to this balance law is supported by classical thermodynamics. Thus, as we see from BAM1 and BAM2 in section 6.3, the balance of angular momenta would yield (using infinitesimal deformation)

$$\epsilon_{mkn}(\sigma_{mk} + S_{mk}) + m_{ln,j} = 0 \quad (160)$$

We note that in (160) ${}_s\boldsymbol{\sigma}$ and ${}_s\mathbf{S} = \mathbf{S}$ are absent due to the presence of permutation tensor. We also note that (160) is purely for nonclassical physics whereas ${}_s\boldsymbol{\sigma}$ and $\mathbf{S}$ are due to classical continuum physics, thus their absence in (160) is natural. In the derivation presented in 6.8.1 we are extracting information related to classical continuum physics from a balance law that is purely derived for nonclassical physics. Following the details of section 6.8.1 and using (160) we will

obtain the following equation containing purely classical continuum physics.

$${}_s\sigma_{mk} + S_{mk} = 0 \quad (161)$$

This should not be a surprise, as (160) that holds for nonclassical physics is not sensitive to the precise relationship between ${}_s\sigma$ and $\mathbf{S}$ that are related to classical physics. At this point we realize that (161) may require modifications to describe the physics related to ${}_s\sigma$ and $\mathbf{S}$. Negative sign for S_{mk} giving ((159) or the following)

$${}_s\sigma_{mk} - S_{mk} = 0 \quad (162)$$

is the correct balance equation for ${}_s\sigma$ and $\mathbf{S}$ between the microconstituents and the medium. The use of identity for BAM3 is in fact motivated by realizing that the physics described by (162) can in fact be derived by using BAM3. We note that balance of angular momenta remain (160) or (163) in case of nonlinear kinematics of the microconstituents.

$$\epsilon_{mkn}(\sigma_{mk}^* + S_{mk}^*) + m_{ln,l}^* = 0 \quad (163)$$

We reiterate that the balance of angular momenta in section 6.3 that is for non-classical physics, BAM1 and BAM2, is totally insensitive to the physics in ${}_s\sigma$ and $\mathbf{S}$ and the relationship between them as these are all due to classical continuum physics. Thus, BAM3 using identity is initiated to recover (162) from the balance of angular momenta. Hence, Equation (162) or (159) is a thermodynamic requirement and not ad hoc imposed condition as it is extracted from a thermodynamic law, balance of angular momenta, BAM3

6.8.3. Consideration for Additional Three Equations

We need additional three equations for the closure of the mathematical model. Before we consider these, let us consider Eringen's microstretch microcontinuum theory. In this theory (as in all microcontinuum theories by Eringen) σ and $\mathbf{m}$ are nonsymmetric tensors and in the constitutive theories for σ and $\mathbf{m}$, these are also considered to be nonsymmetric. In Eringen's theory, we have 29 equations: balance of linear momenta (3), balance of angular momenta (3), energy Equation (1) and constitutive theories $\sigma(9)$, $\mathbf{m}(9)$, $\mathbf{S}(1)$ and $\mathbf{q}(3)$ in 32 dependent variables: $\mathbf{u}(3)$, $\sigma(9)$, $\mathbf{m}(9)$, $\mathbf{J}^{(\alpha)}(4)$, ${}_a\Theta(3)$, $\mathbf{q}(3)$, $\theta(1)$, thus additional three equations are needed for closure of the mathematical model.

Eringen proposes conservation of microinertia as a new conservation law to obtain additional three equations that provides closure to the mathematical model. This mathematical model is currently used in published works. The first comment related to this mathematical model is that balance of momentum and conservation of microinertia are not supported by classical thermodynamics. *i.e.* classical thermodynamics has no such balance and conservation laws. Thus, appending these two laws to actual valid laws of thermodynamics leads to a mathematical framework that is no longer a valid thermodynamic framework. Thus, Eringen's nonlinear microstretch theory is not a valid and thermodynamically inconsistent microcontinuum theory. Constitutive tensors $\mathbf{m}, \sigma$ being nonsymmetric

tensors is in violation of the representation theorem, hence this microcontinuum theory contains nonphysical and mathematically invalid constitutive theories.

Returning back to our quest for obtaining additional three equations needed for closure, we find that the classical thermodynamic framework is unusable to provide any further means of obtaining additional three equations. At this stage we can consider two possibilities.

1) In the first case, we can use Eringen's conservation of microinertia conservation law to obtain additional three equations needed for closure of the mathematical model. The main problem in this approach is that this conservation law is not supported by classical thermodynamics, hence the resulting mathematical model is thermodynamically inconsistent. We point out that in published works this conservation law is routinely used when additional three equations are needed. Our view is that the thermodynamic inconsistency of the resulting theory rules out the theory to be valid microcontinuum theory when this conservation law is used, hence we do not support this approach. If the thermodynamic consistency of the resulting theory is of no concern (as the case is in majority of published works), then we have a mathematical model with closure in which the microconstituents have nonlinear kinematics.

2) In the second approach, we look for an alternative in which the thermodynamic consistency of the resulting theory is preserved and the mathematical model also has closure. Since the requirement of additional nine equations is due to nonlinear kinematics of microconstituents, in this approach we only consider linear microconstituent deformation that requires only one additional equation, thus eliminating the need for additional three equations. In the linear microconstituent kinematics there is only one independent component of ${}^d_s\mathbf{J}^{(\alpha)}$ as microconstituent deformational degrees of freedom. The other three degrees of freedom ${}^c_\Theta^{(\alpha)}$, the classical rotation within the microconstituent volume remain as free field hence not influencing the microconstituent deformation. Thus, now we have a nonlinear microdilation microcontinuum theory in which: the microconstituent kinematics is linear, the solid medium deformation is nonlinear and the interaction of the microconstituents with the solid medium is nonlinear. Another way to rationalize this linear deformation of the microconstituents is to realize that the nonlinear deformation of microconstituents will require very high forces on their surfaces that must be generated by the surrounding medium. This may not be physically possible without generating a very high stress field in the medium that may not be supported by elastic deformation of the solid medium. Perhaps the lack of means in the classical thermodynamics to obtain three additional equations is an indication that nonlinear deformation of microconstituents is not possible in a physical theory supported by classical thermodynamics. Equation (159) clearly shows that trace of ${}_s\mathbf{S}^*$ cannot exceed the trace of ${}_s\boldsymbol{\sigma}^*$, implying that the stresses in the microconstituents are limited by the stresses in the surrounding medium, implying that nonlinear deformation of microconstituents requiring much higher stresses may not be physically possible.

6.8.4. Linear Microconstituent Kinematics

Essentially to eliminate the need for three additional equations, we need to eliminate three degrees of freedom from the microconstituent degrees of freedom ε_v , ${}_c\Theta^{(\alpha)}$, and ${}_c\Theta$, clearly ${}_c\Theta^{(\alpha)}$ is the obvious choice. Let us assume that the microconstituent kinematics is linear, then we have

$${}^dJ^{(\alpha)} = {}^d_sJ^{(\alpha)} + {}^d_aJ^{(\alpha)} = \varepsilon_v\delta + {}^d_aJ^{(\alpha)} \quad (164)$$

$\varepsilon_v\delta$ defines volumetric physics and ${}^d_aJ^{(\alpha)}$ contains three classical rotation ${}_c\Theta^{(\alpha)}$ within the volume of the microconstituent that constitute a free field as the microconstituents are isotropic homogeneous volume of matter that offer no resistance to the rotation field ${}_c\Theta^{(\alpha)}$, thus now the microconstituent deformation and strain measure is completely defined by $\varepsilon^{(\alpha)} = \varepsilon_v\delta$, eliminating the need for additional equations. With this assumption S^* change back to S , symmetric macro Cauchy stress tensor obtained using micro Cauchy stress tensor $\sigma^{(\alpha)}$ through integral-average definition. Balance of angular momenta now contains S (symmetric) instead of S^* , hence S^* is eliminated from it due to permutation tensor and equation (159) yielding additional equation can be modified as

$$tr({}_s\sigma^*) - tr(S) = 0 \quad (165)$$

Likewise the conjugate pairs in the energy equation and entropy inequality containing $S^* : \dot{J}^{(\alpha)}$ are modified to $S : \dot{\varepsilon}^{(\alpha)}$ where

$$\varepsilon^{(\alpha)} = \varepsilon_v\delta \quad (166)$$

The conservation and the balance laws (145)-(150) and the additional equation can now be written as

$$\rho_0(\mathbf{x}) = |\mathbf{J}| \rho(\mathbf{x}, t) \quad (167)$$

$$\rho_0 a_k - \rho_0 ({}^bF_k) - \sigma_{lk,k}^* = 0 \quad (168)$$

$$\epsilon_{mkn} (\sigma_{mk}^* \pm S_{mk}) + m_{ln,l}^* = 0 \quad (169)$$

$$\rho_0 \dot{\varepsilon} - \sigma^* : \dot{J} - S : \dot{\varepsilon}^{(\alpha)} + \nabla \cdot \mathbf{q} - ({}_c\dot{\Theta} \cdot (\nabla \cdot \mathbf{m}^*) + \mathbf{m}^* : {}^c\Theta \dot{J}) \leq 0 \quad (170)$$

$$\rho_0 (\dot{\Phi} + \eta \dot{\theta}) - \sigma^* : \dot{J} - S : \dot{\varepsilon}^{(\alpha)} + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - ({}_c\dot{\Theta} \cdot (\nabla \cdot \mathbf{m}^*) + \mathbf{m}^* : {}^c\Theta \dot{J}) \leq 0 \quad (171)$$

$$\epsilon_{ijk} m_{ij} = 0 \quad (172)$$

$$tr({}_s\sigma^*) - tr(S) = 0 \quad (173)$$

7. Constitutive Theories

In this section, we derive constitutive theory for elastic microdilation microcontinuum in which microconstituents kinematics is linear, solid medium deformation is nonlinear and the interaction of the microconstituents with the solid medium is nonlinear. Initial determination of constitutive tensors and their argument tensors is facilitated using conjugate pairs in the entropy inequality and the

principle of causality [57]. These may be altered and/or augmented depending on upon the desired physics that may not have been considered while deriving entropy inequality. Once, the constitutive tensors and their argument tensors are established that are valid choices based on theory of isotropic tensors, we can use representation theorem to derive mathematically consistent constitutive theories. Material coefficients are established using standard approach based on Taylor series expansion of the coefficient used in the linear combination about a known configuration [56] [57] in the invariants of the argument tensor and temperature θ .

Consider entropy inequality (171)

$$\rho_0 \left(\dot{\Phi} + \eta \dot{\theta} \right) - \boldsymbol{\sigma}^* : \dot{\mathbf{J}} - \mathbf{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \mathbf{m}^* : {}^c \dot{\boldsymbol{\Theta}} \mathbf{J} + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - {}^c \dot{\boldsymbol{\Theta}} \cdot (\nabla \cdot \mathbf{m}^*) \leq 0 \quad (174)$$

In (174), we note that $\boldsymbol{\sigma}^*, \dot{\mathbf{J}}, \mathbf{m}^*, {}^c \dot{\boldsymbol{\Theta}}, {}^c \dot{\boldsymbol{\Theta}} \mathbf{J}$ are not valid measure for finite deformation, finite strain, secondly these are all nonsymmetric tensors, hence representation theorem cannot be used with these as constitutive and argument tensors *i.e.*

$$\begin{aligned} \boldsymbol{\sigma}^* &\neq \boldsymbol{\sigma}^*(\mathbf{J}, \theta), & \mathbf{S} &= \mathbf{S}(\boldsymbol{\varepsilon}^{(\alpha)}, \theta) \text{ is valid} \\ \mathbf{m}^* &\neq \mathbf{m}^*({}^c \boldsymbol{\Theta} \mathbf{J}, \theta) \text{ but } & \mathbf{q} &= \mathbf{q}(\mathbf{g}, \theta) \text{ is valid} \end{aligned} \quad (175)$$

In which $\mathbf{g} = \nabla \theta = \theta_{,i,j}$.

Through additive decomposition, we must express nonsymmetric tensor as sum of symmetric and skew symmetric tensor followed by simplification so that valid conjugate pairs can be established that can be used in representation theorem. Additionally the last term in (174) must also be addressed, we present details in the following.

From balance of angular momenta

$$\nabla \cdot \mathbf{m}^* = -\boldsymbol{\epsilon} : \boldsymbol{\sigma}^* \quad (176)$$

Using (176) in the last term of (174)

$${}^c \dot{\boldsymbol{\Theta}} \cdot (\nabla \cdot \mathbf{m}^*) = -{}^c \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}^*) \quad (177)$$

A simple calculation shows that

$${}^c \dot{\boldsymbol{\Theta}} \cdot (\boldsymbol{\epsilon} : \boldsymbol{\sigma}^*) = {}_a \boldsymbol{\sigma}^* : {}_a \dot{\mathbf{J}} \quad (178)$$

Using (178) in (177)

$${}^c \dot{\boldsymbol{\Theta}} \cdot (\nabla \cdot \mathbf{m}^*) = -{}_a \boldsymbol{\sigma}^* : {}_a \dot{\mathbf{J}} \quad (179)$$

Following reference [1] [53] we have

$$\boldsymbol{\sigma}^* : \dot{\mathbf{J}} = {}_s \boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} + \boldsymbol{\sigma}^* : {}_a \dot{\mathbf{J}} = {}_s \boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} + {}_a \boldsymbol{\sigma}^* : {}_a \dot{\mathbf{J}} \quad (180)$$

Substituting (179) and (180) in (174)

$$\begin{aligned} \rho_0 \left(\dot{\Phi} + \eta \dot{\theta} \right) - {}_s \boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - {}_a \boldsymbol{\sigma}^* : {}_a \dot{\mathbf{J}} - \mathbf{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \mathbf{m}^{[0]} : \left((\mathbf{J}^T) \cdot {}^c \dot{\boldsymbol{\Theta}} \mathbf{J} \right) \\ + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - \left(-{}_a \boldsymbol{\sigma}^* : {}_a \dot{\mathbf{J}} \right) \leq 0 \end{aligned} \quad (181)$$

Or

$$\rho_0 (\dot{\Phi} + \eta \dot{\theta}) - {}_s\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \mathbf{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \mathbf{m}^{[0]} : \left((\mathbf{J}^T) \cdot {}^{c^\Theta} \mathbf{J} \right) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} \leq 0 \quad (182)$$

We further note that the volumetric deformation and the distortional deformation are mutually exclusive, hence cannot be described by a single constitutive theory for ${}_s\boldsymbol{\sigma}^{[0]}$. Thus, we must consider additive decomposition of ${}_s\boldsymbol{\sigma}^{[0]}$ into equilibrium and deviatoric stress tensors (${}_s^e\boldsymbol{\sigma}^{[0]}, {}_s^d\boldsymbol{\sigma}^{[0]}$). Constitutive theory for ${}_s^e\boldsymbol{\sigma}^{[0]}$ addresses volumetric deformation and the constitutive for ${}_s^d\boldsymbol{\sigma}^{[0]}$ describes distortional deformation.

$${}_s\boldsymbol{\sigma}^{[0]} = {}_s^e\boldsymbol{\sigma}^{[0]} + {}_s^d\boldsymbol{\sigma}^{[0]} \quad (183)$$

Using (183) in (182)

$$\rho_0 (\dot{\Phi} + \eta \dot{\theta}) - {}_s^e\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - {}_s^d\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \mathbf{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \mathbf{m}^{[0]} : \left(\mathbf{J}^T \cdot ({}^{c^\Theta} \mathbf{J}) \right) \leq 0 \quad (184)$$

Since $\mathbf{m}^{[0]}$ is symmetric, the last term in (184) can be simplified

$$\mathbf{m}^{[0]} : \left(\mathbf{J}^T \cdot ({}^{c^\Theta} \mathbf{J}) \right) = \frac{1}{2} \mathbf{m}^{[0]} : \left(\mathbf{J}^T \cdot ({}^{c^\Theta} \mathbf{J}) + ({}^{c^\Theta} \mathbf{J})^T \cdot \mathbf{J} \right) \quad (185)$$

$$\mathbf{m}^{[0]} : \left(\mathbf{J}^T \cdot ({}^{c^\Theta} \mathbf{J}) \right) = \mathbf{m}^{[0]} : {}^{c^\Theta} \dot{\boldsymbol{\varepsilon}}_{[0]} \quad (186)$$

Where

$$\begin{aligned} {}^{c^\Theta} \dot{\boldsymbol{\varepsilon}}_{[0]} &= \frac{1}{2} \left(\mathbf{J}^T \cdot ({}^{c^\Theta} \mathbf{J}) + ({}^{c^\Theta} \mathbf{J})^T \cdot \mathbf{J} \right) \\ &= \frac{1}{2} \left(({}_s\mathbf{J} + {}_a\mathbf{J}) \cdot ({}_s^c\mathbf{J} + {}_a^c\mathbf{J}) + ({}_s^c\mathbf{J} + {}_a^c\mathbf{J}) \cdot ({}_s\mathbf{J} + {}_a\mathbf{J}) \right) \\ &= ({}_s\mathbf{J} \cdot {}_s^c\mathbf{J} + {}_a\mathbf{J} \cdot {}_a^c\mathbf{J}) \end{aligned} \quad (187)$$

Now we can rewrite (184) using (187)

$$\rho_0 (\dot{\Phi} + \eta \dot{\theta}) - {}_s^e\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - {}_s^d\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \mathbf{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \mathbf{m}^{[0]} : ({}^{c^\Theta} \dot{\boldsymbol{\varepsilon}}_{[0]}) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} \leq 0 \quad (188)$$

where ${}^{c^\Theta} \dot{\boldsymbol{\varepsilon}}_{[0]}$ is given by (187).

In the entropy inequality (188) all tensors of rank two in the rate of work conjugate pairs are symmetric tensors, hence conjugate pairs in (188) are suitable for representation theorem. All conjugate pairs in (188) are good.

We use (188) to establish constitutive tensors and their possible argument tensors using the conjugate pairs in (188) in conjunction with axiom of causality [57].

$${}_s^e\boldsymbol{\sigma}^{[0]} = {}_s^e\boldsymbol{\sigma}^{[0]}(\rho, \theta) \quad (189)$$

$${}_s^d\boldsymbol{\sigma}^{[0]} = {}_s^d\boldsymbol{\sigma}^{[0]}(\boldsymbol{\varepsilon}_{[0]}, \theta) \quad (190)$$

$$\mathbf{S} = \mathbf{S}(\boldsymbol{\varepsilon}^{(\alpha)}, \theta) \quad (191)$$

$$\mathbf{m}^{[0]} = \mathbf{m}^{[0]}({}^{c^\Theta} \boldsymbol{\varepsilon}_{[0]}, \theta) \quad (192)$$

$$\mathbf{q} = \mathbf{q}(\mathbf{g}, \theta) \quad (193)$$

Even though we do not require constitutive theory for Φ and η , their argument tensors are necessary as these are used to simplify entropy inequality (188). Except θ , we do not really have a mechanism to choose the argument tensor of Φ and η , so we use principle of equipresence.

$$\Phi = \Phi(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}^{(\alpha)}, {}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}, \mathbf{g}, \theta) \quad (194)$$

$$\eta = \eta(\boldsymbol{\varepsilon}_{[0]}, \boldsymbol{\varepsilon}^{(\alpha)}, {}^{c\Theta}\boldsymbol{\varepsilon}_{[0]}, \mathbf{g}, \theta) \quad (195)$$

From the physics of pure volumetric deformation, we know that equilibrium stress must be a function of density and temperature, hence the argument tensor in (189) to emphasize dependence of ${}_s\boldsymbol{\sigma}^{[0]}$ on ρ (symbolic) and θ , even though ρ is not admissible as an argument tensor in Lagrangian description.

7.1. Constitutive Theory for ${}_s\boldsymbol{\sigma}^{[0]}$

Compressibility, hence density in solids is controlled by $|\mathbf{J}|$ and the density in the current configuration is deterministic from conservation of mass when $\mathbf{J}$ is known. Thus, density is not a dependent variable in the conservation and balance laws in Lagrangian description for solid matter. The equation of state in solid matter is a consequence of density change caused due to $|\mathbf{J}|$, and there is a pressure field associated with the density change. The presence of this pressure field through equilibrium stress tensor in the balance of linear momenta is essential for correct force balance. Thus, in compressible solid matter one could determine evolution using the mathematical model without using equation of state. But, such solution would be in error due to incorrect force balance in the balance of linear momenta. Since the compressibility physics depends upon density and temperature the constitutive theory for ${}_s\boldsymbol{\sigma}^{[0]}$ must be obtained using the constitutive theory for ${}^e\boldsymbol{\sigma}^{(0)}$, the equilibrium Cauchy stress tensor. Details of this derivation can be found in a recent paper [1] [53] and reference [57]. The final form of the constitutive theory for ${}_s\boldsymbol{\sigma}^{[0]}$ for compressible and incompressible non-isothermal physics for micro-dilation solid media are given by

$${}_s\boldsymbol{\sigma}^{[0]} = |\mathbf{J}|(\mathbf{J}^{-1})p(\rho, \theta)\boldsymbol{\delta}(\mathbf{J}^{-1})^T = |\mathbf{J}|p(\rho, \theta)\boldsymbol{\delta}(\mathbf{J}^T \cdot \mathbf{J})^{-1} \text{ Compressible} \quad (196)$$

$${}_s\boldsymbol{\sigma}^{[0]} = |\mathbf{J}|(\mathbf{J}^{-1})p(\theta)\boldsymbol{\delta}(\mathbf{J}^{-1})^T = |\mathbf{J}|p(\theta)\boldsymbol{\delta}(\mathbf{J}^T \cdot \mathbf{J})^{-1} \text{ Incompressible} \quad (197)$$

In which $p(\rho, \theta)$ and $p(\theta)$ are thermodynamic and mechanical pressures. In (197) we could have used $\mathbf{J} \approx \mathbf{I}$ and $|\mathbf{J}| \approx 1$, but we leave the expression as is in (197) for generality.

The reduced form of the entropy inequality (188) (after deriving constitutive theory for ${}_s\boldsymbol{\sigma}^{[0]}$) in Lagrangian description can be rewritten as

$$\frac{\mathbf{q} \cdot \mathbf{g}}{\theta} - {}^d{}_s\boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \mathbf{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \mathbf{m}^{[0]} : ({}^{c\Theta}\dot{\boldsymbol{\varepsilon}}_{[0]}) \leq 0 \quad (198)$$

7.2. Constitutive Theory for ${}^d_s\boldsymbol{\sigma}^{[0]}$

Constitutive theory for ${}^d_s\boldsymbol{\sigma}^{[0]}$ must address distortional deformation physics of the medium (without volumetric change). Using (190) with $\boldsymbol{\varepsilon}_{[0]}$ and θ as argument tensor of ${}^d_s\boldsymbol{\sigma}^{[0]}$, we can derive the constitutive theory for ${}^d_s\boldsymbol{\sigma}^{[0]}$ using representation theorem.

Let ${}^\sigma\boldsymbol{G}^i; i=1,2,\dots,N^\sigma$ be the combined generators of the argument tensors of ${}^d_s\boldsymbol{\sigma}^{[0]}$ in (190) that are symmetric tensor of rank two. Then, $\boldsymbol{I}, {}^\sigma\boldsymbol{G}^i; i=1,2,\dots,N^\sigma$ constitute the basis (integrity) of the space of tensor ${}^d_s\boldsymbol{\sigma}^{[0]}$. Thus, ${}^d_s\boldsymbol{\sigma}^{[0]}$ can be expressed as a linear combination of the basis (in the current configuration).

$${}^d_s\boldsymbol{\sigma}^{[0]} = {}^\sigma\alpha^0 \boldsymbol{I} + \sum_{i=1}^{N^\sigma} {}^\sigma\alpha^i ({}^\sigma\boldsymbol{G}^i) \quad (199)$$

in which

$${}^\sigma\alpha^i = {}^\sigma\alpha^i({}^\sigma\boldsymbol{I}^j, \theta); i=0,1,\dots,N^\sigma; j=1,2,\dots,M^\sigma \quad (200)$$

and ${}^\sigma\boldsymbol{I}^j; j=1,2,\dots,M^\sigma$ are the combined invariants of the argument tensors of ${}^d_s\boldsymbol{\sigma}^{[0]}$ in (190).

The material coefficients in (199) are determined by expanding ${}^\sigma\alpha^i; i=0,1,\dots,N^\sigma$ in Taylor series in ${}^\sigma\boldsymbol{I}^j; j=1,2,\dots,M^\sigma$ and θ about a known configuration $\underline{\Omega}$ and only retaining up to linear terms in ${}^\sigma\boldsymbol{I}^j; j=1,2,\dots,M^\sigma$ and θ (for the sake of simplicity of the resulting theory).

$${}^\sigma\alpha^i = {}^\sigma\alpha^i|_{\underline{\Omega}} + \sum_{j=1}^{M^\sigma} \frac{\partial({}^\sigma\alpha^i)}{\partial({}^\sigma\boldsymbol{I}^j)} \Big|_{\underline{\Omega}} ({}^\sigma\boldsymbol{I}^j - {}^\sigma\boldsymbol{I}^j|_{\underline{\Omega}}) + \frac{\partial({}^\sigma\alpha^i)}{\partial\theta} \Big|_{\underline{\Omega}} (\theta - \theta|_{\underline{\Omega}}); \quad (201)$$

$$i=0,1,\dots,N^\sigma$$

Substituting ${}^\sigma\alpha^0$ and ${}^\sigma\alpha^i; i=1,2,\dots,N^\sigma$ in (199)

$$\begin{aligned} {}^d_s\boldsymbol{\sigma}^{[0]} = & \left({}^\sigma\alpha^0|_{\underline{\Omega}} + \sum_{j=1}^{M^\sigma} \frac{\partial({}^\sigma\alpha^0)}{\partial({}^\sigma\boldsymbol{I}^j)} \Big|_{\underline{\Omega}} ({}^\sigma\boldsymbol{I}^j - {}^\sigma\boldsymbol{I}^j|_{\underline{\Omega}}) + \frac{\partial({}^\sigma\alpha^0)}{\partial\theta} \Big|_{\underline{\Omega}} (\theta - \theta|_{\underline{\Omega}}) \right) \boldsymbol{I} \\ & + \sum_{i=1}^{N^\sigma} \left({}^\sigma\alpha^i|_{\underline{\Omega}} + \sum_{j=1}^{M^\sigma} \frac{\partial({}^\sigma\alpha^i)}{\partial({}^\sigma\boldsymbol{I}^j)} \Big|_{\underline{\Omega}} ({}^\sigma\boldsymbol{I}^j - {}^\sigma\boldsymbol{I}^j|_{\underline{\Omega}}) + \frac{\partial({}^\sigma\alpha^i)}{\partial\theta} \Big|_{\underline{\Omega}} (\theta - \theta|_{\underline{\Omega}}) \right) {}^\sigma\boldsymbol{G}^i \end{aligned} \quad (202)$$

Collecting coefficients of $\boldsymbol{I}, {}^\sigma\boldsymbol{I}^j \boldsymbol{I}, {}^\sigma\boldsymbol{G}^i, {}^\sigma\boldsymbol{I}^j {}^\sigma\boldsymbol{G}^i, (\theta - \theta|_{\underline{\Omega}}) {}^\sigma\boldsymbol{G}^i$ and $(\theta - \theta|_{\underline{\Omega}}) \boldsymbol{I}$ in (202), we can write (202) as follows:

$$\begin{aligned} {}^d_s\boldsymbol{\sigma} = & \sigma_0 \boldsymbol{I} + \sum_{j=1}^{M^\sigma} {}^\sigma a_j ({}^\sigma\boldsymbol{I}^j) \boldsymbol{I} + \sum_{i=1}^{N^\sigma} {}^\sigma b_i ({}^\sigma\boldsymbol{G}^i) + \sum_{j=1}^{M^\sigma} \sum_{i=1}^{N^\sigma} {}^\sigma c_{ij} ({}^\sigma\boldsymbol{I}^j) ({}^\sigma\boldsymbol{G}^i) \\ & - \sum_{i=1}^{N^\sigma} {}^\sigma d_i (\theta - \theta|_{\underline{\Omega}}) ({}^\sigma\boldsymbol{G}^i) - {}^\sigma\alpha_{tm} (\theta - \theta|_{\underline{\Omega}}) \boldsymbol{I} \end{aligned} \quad (203)$$

The material coefficients ${}^\sigma a_j, {}^\sigma b_i, {}^\sigma c_{ij}, {}^\sigma d_i$ and ${}^\sigma\alpha_{tm}$ are defined in the following:

$$\begin{aligned}
\sigma_0 &= {}^\sigma \alpha^0 \Big|_{\underline{\Omega}} - \sum_{j=1}^{M^\sigma} \frac{\partial({}^\sigma \alpha^0)}{\partial({}^\sigma \underline{I}^j)} \Big|_{\underline{\Omega}} \left(-{}^\sigma \underline{I}^j \Big|_{\underline{\Omega}} \right) & {}^\sigma q_j &= \frac{\partial({}^\sigma \alpha^0)}{\partial({}^\sigma \underline{I}^j)} \Big|_{\underline{\Omega}} \\
{}^\sigma \underline{b}_i &= {}^\sigma \alpha^i \Big|_{\underline{\Omega}} + \sum_{j=1}^{M^\sigma} \frac{\partial({}^\sigma \alpha^i)}{\partial({}^\sigma \underline{I}^j)} \Big|_{\underline{\Omega}} \left(-{}^\sigma \underline{I}^j \Big|_{\underline{\Omega}} \right) & {}^\sigma \underline{c}_{ij} &= \frac{\partial({}^\sigma \alpha^i)}{\partial({}^\sigma \underline{I}^j)} \Big|_{\underline{\Omega}} \\
{}^\sigma \underline{d}_i &= -\frac{\partial({}^\sigma \alpha^i)}{\partial \theta} \Big|_{\underline{\Omega}} & {}^\sigma \alpha_{im} &= -\frac{\partial({}^\sigma \alpha^0)}{\partial \theta} \Big|_{\underline{\Omega}}
\end{aligned} \quad (204)$$

The constitutive theory (203) is based on integrity (complete basis of the space of tensor ${}^d_s \boldsymbol{\sigma}^{[0]}$ and requires $\left(2N^\sigma + (M^\sigma)(N^\sigma) + M^\sigma + 1\right)$ material coefficient. Various simplified forms of the constitutive theories can be derived from (203) by choosing desired generators and invariants. Based on (190) in this constitutive theory $N^\sigma = 2$, $M^\sigma = 3$ and ${}^\sigma \underline{\mathbf{G}}^1 = \boldsymbol{\varepsilon}_{[0]}$, ${}^\sigma \underline{\mathbf{G}}^2 = \boldsymbol{\varepsilon}_{[0]}^2$ and the three invariants are $I_{\varepsilon_{[0]}}, II_{\varepsilon_{[0]}}, III_{\varepsilon_{[0]}}$.

Most simplified form of the constitutive theory is obtained for $N^\sigma = 1$ (after redefining material coefficient).

$${}^d_s \boldsymbol{\sigma} = \sigma_0 \mathbf{I} + 2\mu^\sigma \left(\boldsymbol{\varepsilon}_{[0]} \right) + \lambda^\sigma \left(\text{tr} \left(\boldsymbol{\varepsilon}_{[0]} \right) \right) \mathbf{I} - {}^\sigma \alpha_{im} \left(\theta - \theta \Big|_{\underline{\Omega}} \right) \mathbf{I} \quad (205)$$

The material coefficient could be function of all three invariant and θ in a known configuration $\underline{\Omega}$. μ^σ and λ^σ are similar to Lames constant in linear classical elasticity.

7.3. Constitutive Theory for $\mathbf{S}$

Consider (191) *i.e.*

$$\mathbf{S} = \mathbf{S} \left(\boldsymbol{\varepsilon}^{(\alpha)}, \theta \right) \quad (206)$$

In which $\boldsymbol{\varepsilon}^{(\alpha)}$ is the strain tensor for the microconstituents. Let ${}^s \underline{\mathbf{G}}^i; i=1, 2, \dots, N^s$ be the combined generators of the argument tensors of $\mathbf{S}$ in (206) and let ${}^s \underline{I}^j; j=1, 2, \dots, M^s$ be the combined invariants of the same argument tensors of $\mathbf{S}$ in (206), then $\mathbf{I}, {}^s \underline{\mathbf{G}}^i; i=1, 2, \dots, N^s$ constitutes the basis of the space of constitutive tensor $\mathbf{S}$, hence we can express $\mathbf{S}$ in a linear combination of the basis (Integrity).

$$\mathbf{S} = {}^s \alpha^0 \mathbf{I} + \sum_{i=1}^{N^s} {}^s \alpha^i \left({}^s \underline{\mathbf{G}}^i \right) \quad (207)$$

In which the coefficient ${}^s \alpha^i; i=0, 1, \dots, N^s$ in the linear combination (207) are functions of ${}^s \underline{I}^j; j=1, 2, \dots, M^s$ and temperature θ .

Material coefficient in (207) are determine using exactly same approach as described and used for ${}^d_s \boldsymbol{\sigma}^{[0]}$ in section 7.2.

Expanding ${}^s \alpha^i; i=0, 1, \dots, N^s$ in Taylor series in ${}^s \underline{I}^j; j=1, 2, \dots, M^s$ and θ about a known configuration $\underline{\Omega}$ and retaining only up to linear terms in ${}^s \underline{I}^j; j=1, 2, \dots, M^s$ and θ , substituting these in (207) and collecting coefficient of $\mathbf{I}, {}^s \underline{I}^j \mathbf{I}, {}^s \underline{\mathbf{G}}^i, ({}^s \underline{I}^j)({}^s \underline{\mathbf{G}}^i), (\theta - \theta \Big|_{\underline{\Omega}})({}^s \underline{\mathbf{G}}^i)$ and $(\theta - \theta \Big|_{\underline{\Omega}}) \mathbf{I}$ we can obtain

$$\begin{aligned} \mathbf{S} = & S_0 \mathbf{I} + \sum_{j=1}^{M^s} {}^s a_j \left({}^s \underline{I}^j \right) \mathbf{I} + \sum_{i=1}^{N^s} {}^s b_i \left({}^s \underline{G}^i \right) + \sum_{j=1}^{M^s} \sum_{i=1}^{N^s} {}^s c_{ij} \left({}^s \underline{I}^j \right) \left({}^s \underline{G}^i \right) \\ & - \sum_{i=1}^{N^s} {}^s d_i \left(\theta - \theta|_{\underline{\Omega}} \right) \left({}^s \underline{G}^i \right) - {}^s \alpha_{im} \left(\theta - \theta|_{\underline{\Omega}} \right) \mathbf{I} \end{aligned} \quad (208)$$

The Taylor series expansion of ${}^s \alpha^i; i = 0, 1, \dots, N^s$ can be obtained from (201) by replacing ${}^\sigma \alpha^i$ with ${}^s \alpha^i$, ${}^\sigma \underline{I}^j$ with ${}^s \underline{I}^j$, N^σ and M^σ with N^s and M^s . The material coefficient ${}^s a_j, {}^s b_i, {}^s c_{ij}, {}^s d_i, {}^s \alpha_{im}$ can be obtained using (204) by replacing ${}^\sigma \alpha^i$ with ${}^s \alpha^i$; ${}^\sigma \underline{I}^j$ with ${}^s \underline{I}^j$; N^σ and M^σ with N^s and M^s . Also ${}^\sigma \sigma_0, {}^\sigma a_j, {}^\sigma b_i, {}^\sigma c_{ij}, {}^\sigma d_i, {}^\sigma \alpha_{im}$ are replaced by $S_0, {}^s a_j, {}^s b_i, {}^s c_{ij}, {}^s d_i$ and ${}^s \alpha_{im}$. The material coefficient can be function of ${}^s \underline{I}^j|_{\underline{\Omega}}; j = 1, 2, \dots, M^s$ and $\theta|_{\underline{\Omega}}$.

The constitutive theory (208) is based on integrity (complete basis of the space of $\mathbf{S}$). Simplified form of (208) can be obtained by choosing desired generators and its invariants. A constitutive theory for $\mathbf{S}$ that is linear in the components of $\boldsymbol{\varepsilon}^{(\alpha)}$ and θ is given by

$$\mathbf{S} = S_0 \mathbf{I} + 2\mu^s \left(\boldsymbol{\varepsilon}^{(\alpha)} \right) + \lambda^s \left(\text{tr} \left(\boldsymbol{\varepsilon}^{(\alpha)} \right) \right) \mathbf{I} - {}^s \alpha_{im} \left(\theta - \theta|_{\underline{\Omega}} \right) \mathbf{I} \quad (209)$$

Remarks

1) $\mathbf{S}$ and $\boldsymbol{\varepsilon}^{(\alpha)}$ are diagonal tensors,

$$\mathbf{S} = s\boldsymbol{\delta}; \boldsymbol{\varepsilon}^{(\alpha)} = \varepsilon^{(\alpha)}\boldsymbol{\delta} \quad (210)$$

2) In this case:

$$\begin{aligned} {}^s \underline{G}^1 &= \varepsilon_v \boldsymbol{\delta}, \quad {}^s \underline{G}^2 = \varepsilon_v^2 \boldsymbol{\delta} \\ I_{\varepsilon_v} &= 3\varepsilon_v; \quad II_{\varepsilon_v} = 3\varepsilon_v^2; \quad III_{\varepsilon_v} = 3\varepsilon_v^3 \end{aligned} \quad (211)$$

7.4. Constitutive Theory for Moment Tensor $\mathbf{m}^{[0]}$

Consider Equation (192)

$$\mathbf{m}^{[0]} = \mathbf{m}^{[0]} \left({}^c \Theta \boldsymbol{\varepsilon}_{[0]}, \theta \right) \quad (212)$$

Let ${}^m \underline{G}^i; i = 1, 2, \dots, N^m$ be the combined generators of the argument tensors of $\mathbf{m}^{[0]}$ in (212) that are symmetric tensor of rank two and let ${}^m \underline{I}^j; j = 1, 2, \dots, M^m$ be the combined invariants of the same argument tensors of $\mathbf{m}^{[0]}$ in (212). Then $\mathbf{I}, {}^m \underline{G}^i; i = 1, 2, \dots, N^m$ constitutes the basis of the space of tensor $\mathbf{m}^{[0]}$, hence we can represent $\mathbf{m}^{[0]}$ as a linear combination of the basis (integrity).

$$\mathbf{m}^{[0]} = {}^m \alpha^0 \mathbf{I} + \sum_{i=1}^{N^m} {}^m \alpha^i \left({}^m \underline{G}^i \right) \quad (213)$$

In which the coefficients ${}^m \alpha^i; i = 0, 1, \dots, N^m$ in the linear combination can be functions of ${}^m \underline{I}^j; j = 1, 2, \dots, M^m$, the combined invariants of the argument tensor in (212) and temperature θ . The material coefficient in (213) are determined using the approach used in section 7.2 for ${}^d \sigma^{[0]}$. We expand ${}^m \alpha^i; i = 0, 1, \dots, N^m$ in Taylor series in ${}^m \underline{I}^j; j = 1, 2, \dots, M^m$ and θ about a known configuration $\underline{\Omega}$ and retain only up to linear terms in ${}^m \underline{I}^j; j = 1, 2, \dots, M^m$ and θ . The resulting expression can be obtained from (201) by replacing ${}^\sigma \alpha^i$ with ${}^m \alpha^i$; ${}^\sigma \underline{I}^j$ with

${}^m \underline{I}^j$; N^σ and M^σ with N^m and M^m . Substituting these ${}^m \alpha^i; i = 0, 1, \dots, N^m$ in (213) and collecting coefficients of $I, {}^m \underline{I}^j I, {}^m \underline{G}^i, ({}^m \underline{I}^j)({}^m \underline{G}^i), (\theta - \theta|_\Omega)({}^m \underline{G}^i)$ and $(\theta - \theta|_\Omega)I$ we can write the following.

$$\begin{aligned} \mathbf{m}^{[0]} = & m_0 I + \sum_{j=1}^{M^m} {}^m a_j ({}^m \underline{I}^j) I + \sum_{i=1}^{N^m} {}^m b_i ({}^m \underline{G}^i) + \sum_{i=1}^{N^m} \sum_{j=1}^{M^m} {}^m c_{ij} ({}^m \underline{I}^j) ({}^m \underline{G}^i) \\ & - \sum_{i=1}^{N^m} {}^m d_i (\theta - \theta|_\Omega) {}^m \underline{G}^i - {}^m \alpha_m (\theta - \theta|_\Omega) I \end{aligned} \quad (214)$$

The material coefficient ${}^m a_j, {}^m b_i, {}^m c_{ij}, {}^m d_i$ and ${}^m \alpha_m$ as well as m_0 can be obtained by replacing N^σ, M^σ with N^m, M^m , σ_0 with m_0 and ${}^\sigma a_j, {}^\sigma b_i, {}^\sigma c_{ij}, {}^\sigma d_i$ with ${}^m a_j, {}^m b_i, {}^m c_{ij}, {}^m d_i$ and ${}^\sigma \alpha_m$ by ${}^m \alpha_m$. The constitutive theory (214) is based on complete basis (integrality) of the space of constitutive tensor $\mathbf{m}^{[0]}$. Simplified form of the constitutive theories for $\mathbf{m}^{[0]}$ can be obtained from (212) by retaining desired generators and the invariants. A constitutive theory that is linear in the components of the argument tensor is given by (after re-defining material coefficients)

$$\mathbf{m}^{[0]} = m_0 I + 2\mu^m ({}^c \Theta \mathcal{E}_{[0]}) + \lambda^m \left(\text{tr} ({}^c \Theta \mathcal{E}_{[0]}) \right) I - {}^m \alpha_m (\theta - \theta|_\Omega) I \quad (215)$$

7.5. Constitutive Theory for $\mathbf{q}$

Consider

$$\mathbf{q} = \mathbf{q}(\mathbf{g}, \theta) \quad (216)$$

Following references [56] [57], we can derive the following constitutive theory for $\mathbf{q}$ using representation theorem.

$$\mathbf{q} = -\kappa \mathbf{g} - \kappa_1 (\mathbf{g} \cdot \mathbf{g}) \mathbf{g} - \kappa_2 (\theta - \theta|_\Omega) \mathbf{g} \quad (217)$$

κ, κ_1 and κ_2 are material coefficients. These can be functions of $(\mathbf{g} \cdot \mathbf{g})|_\Omega$ and $\theta|_\Omega$. In (217) $\mathbf{g} \cdot \mathbf{g}$ is invariant of argument tensors of $\mathbf{q}$. Simplified form of (217), the Fourier heat conduction law is given by

$$\mathbf{q} = -\kappa \mathbf{g} \quad (218)$$

8. Thermodynamic and Mathematical Consistency of the Microdilation Theory Presented in This Paper

The laws of classical thermodynamics used in classical continuum mechanics are well-founded and accepted laws. Microcontinuum theories contain new physics beyond classical continuum mechanics, hence may require new considerations. For establishing conservation and balance laws for microcontinuum theories in general, we must begin with classical thermodynamics, but can only make changes in them and incorporate new conservation and balance laws if the classical thermodynamics framework supports these. The resulting microcontinuum theory will be referred to as thermodynamically consistent with the law of classical thermodynamics *i.e.* classical continuum mechanics. We list important features of the

present work that establish thermodynamical and mathematical consistency of the nonlinear microdilation theory presented in this paper.

1) It is shown that if we consider nonlinear deformation of the microconstituents, then microconstituent classical rotation $\epsilon \otimes^{(\alpha)}$ are three additional unknowns in the theory that require three additional equations for closure. The classical thermodynamics has no provision for obtaining these from the existing balance laws and also does not provide any means of deriving them otherwise. We could consider Eringen's conservation of microinertia to obtain three additional equations. However this conservation law is not supported by classical thermodynamics, hence the resulting theory would be thermodynamically inconsistent, thus an invalid theory. We do not advocate this approach for obtaining closure to the mathematical model. Thus, we are left with no choice but to consider linear kinematics of the microconstituents in which case these additional three equations are not needed and we have a thermodynamically consistent theory.

2) When the microconstituent deformation is nonlinear, Eringen advocated using conservation of microinertia as a conservation law to obtain additional equations. We have discussed that use of this conservation and balance law is not supported by classical thermodynamics framework, hence the resulting theory is thermodynamically inconsistent. However, in view of the fact that all published works on microstretch theories use this conservation law, we have presented details of the rate of work conjugate pair in Appendix A when the microconstituent deformation is nonlinear. With this rate of work conjugate pair it is straight forward to derive constitutive theories for nonlinear microconstituent kinematics. This will serve as consistent mathematical model that can be used with conservation of microinertia conservation law to obtain closure to the mathematical model.

3) Existence of moment independent of forces that is conjugate to rotations is a result of the resistance offered by the medium to the rigid rotations of the microconstituents. Balance of angular momenta, a statement of balance of moments (of forces in classical continuum mechanics) permits inclusion of the moment tensor in the balance of angular momenta. Thus, this modification of the balance law of classical thermodynamics, balance of angular momenta is supported by classical thermodynamics.

4) In classical thermodynamics, a kinematically conjugate pair requires two balance laws. Kinematically conjugate pair of displacements and forces require two balance laws: balance of forces and balance of moment of forces *i.e.* balance of linear momenta and balance of angular momenta. Based on this, the classical thermodynamics will permit two additional balance laws for each new kinematically conjugate pair. Thus, for the kinematically conjugate pair of rotations and moments in the microcontinuum theories, we need two new balance laws: balance of moments which already exists as balance of angular momenta and can be modified to include moment tensor as discussed in (1) and balance of moment of moments which is a new balance law needed in the microcontinuum theories. Consequence of this balance law is that Cauchy moment tensor becomes symmetric.

In the absence of this balance law dynamic equilibrium of moment of moments is violated, hence thermodynamic consistency is violated.

5) It has been shown by Surana *et al.* that if classical rotations are not used as rigid rotations of the microconstituents, entropy inequality is violated. That is, a microcontinuum theory based $\alpha \Theta$ as unknown rigid rotations of the microconstituents or $\epsilon \Theta + \alpha \Theta$ as rigid rotations of the microconstituents results in violation of entropy inequality. These choices produce additional terms in the entropy inequality that cannot be accounted for, thus resulting in the violation of thermodynamic consistency of the theory.

6) Since rotations and moments are a new kinematically conjugate pair in microcontinuum theories that does not exist in classical continuum mechanics, therefore, the integral-average definition of moment tensor cannot be derived using microconstituent Cauchy stress tensor $\bar{\sigma}^{(\alpha)}$ or $\sigma^{(\alpha)}$ as this stress is due to classical continuum mechanics. Insistence in doing so will result in a theory that is thermodynamically inconsistent.

7) In micropolar microcontinuum theories, (1)-(4) that are supported by classical thermodynamics are sufficient to yield a microcontinuum theory that is thermodynamically consistent and has closure when the constitutive theories are included.

8) When the microconstituents are deformable, (1)-(4) are not sufficient (along with constitutive theories) to provide closure to the mathematical model. In case of microdilation theory, one additional equation is needed for closure. We have shown that balance of angular momenta in fact contains this equation but it is eliminated due to presence of permutation tensor with the stress terms. We have shown that by premultiplying balance of angular momenta with the inverse of the permutation tensor and by taking trace of the resulting equation we can recover the additional equations needed for closure. This part of the derivation is related to balance of angular momenta, hence obviously does not violate thermodynamic consistency.

9) Thus, we note that the use of (1)-(4) or (1)-(4) and (6) that are supported by classical thermodynamics yield conservation and balance laws of all microcontinuum theories, confirming that the conservation and the balance laws in these theories derived using the approach presented in this paper are thermodynamically consistent.

10) In case of constitutive theories, we must use conjugate pairs in the entropy inequality and axiom of causality to determine constitutive tensors and their argument tensors that are supported by the theory of isotropic tensors (as done in the present work). A violation of this results in thermodynamic inconsistency as well as mathematical inconsistency of the resulting theory.

11) Constitutive theories must be derived strictly using representation theorem (as done in the present work) to ensure mathematical consistency of the resulting constitutive theories. If the constitutive theories are derived using any other means such as potentials and energy functionals, then we must show that the same

theories can also be derived using representation theorem, otherwise the constitutive theories derived without using representation theorem are mathematically inconsistent. Clearly the constitutive theories presented in the paper are mathematically and thermodynamically consistent.

12) The two new conservation and the balance laws introduced by Eringen: (1) Conservation of microinertia and (2) balance of moment of symmetric parts of the stress tensors with the gradients of the symmetric part of the moment tensor are not supported by the classical thermodynamics *i.e.* classical continuum mechanics, hence can only be viewed as phenomenological or ad-hoc. Inclusion of these in the laws of classical thermodynamics used in deriving conservation and the balance laws for microcontinuum theories will result in a thermodynamically inconsistent microcontinuum theory.

9. Microcontinuum Theories of Eringen

We summarize some aspects of Eringen's microcontinuum theories that have lead to their thermodynamic and mathematical inconsistencies. These are applicable to microcontinuum theories in general, hence also hold for the nonlinear microdilation theory presented in this paper.

1) Use of $\alpha \otimes$ or $\alpha \otimes + c \otimes$ as rigid rotations of the microconstituents results in violation of entropy inequality, hence thermodynamic inconsistency of the resulting theory.

2) Including rigid rotations in the strain measures in Eringen's work results in tensors that cannot be used in the constitutive theories without violating physics of deformation.

3) Eringen's work defines integral-average moment tensor (nonclassical physics) using microconstituent Cauchy stress tensor $\bar{\sigma}^{(\alpha)}$ or $\sigma^{(\alpha)}$ that is due to classical continuum mechanics. This is obviously wrong. The origin of moment is due to resistance offered to the rigid rotations of the microconstituents by the medium and not $\sigma^{(\alpha)}$. Due to this wrong definition the balance laws such as balance of angular momenta that uses this moment tensor is of concern.

4) Use of weighted integral of balance of micro linear momenta using a weight function $\tilde{\phi}^{(\alpha)}(\bar{x}_m^{(\alpha)})$ with three different choices for balance of linear momenta, balance of angular momenta and the new balance law proposed has no thermodynamic foundation. Our work in the paper shows that this is neither needed nor used.

5) Use of nonsymmetric tensors of rank two as constitutive tensors and the nonsymmetric tensors of rank two as their argument tensor is not supported by the theory of isotropic tensor. It results in constitutive theories that are mathematical inconsistent and are nonphysical.

6) Constitutive theories derived using potentials or energy functional (as in Eringen's work) are nonphysical, not valid and mathematically inconsistent if the same theories cannot be derived using representation theorem.

7) Due to not using balance of moment of moments balance law, the dynamic

equilibrium is not satisfied in the Eringen's mathematical model. Another consequence of not using this balance law is that moment tensor is non-symmetric resulting in spurious constitutive theories.

8) Use of principle of equipresence introduces nonphysical coupling between classical and nonclassical physics and results in nonphysical material coefficients.

9) Lack of various additive decompositions of the stress tensors leads to non-physical and non-valid constitutive tensors. For example ${}_a\sigma$ must be eliminated from σ as it is defined by balance of angular momenta hence cannot be part of constitutive theory. Further decomposition of ${}_s\sigma = {}^e{}_s\sigma + {}^d{}_s\sigma$ is necessary to address volumetric and distortional physics correctly as these are mutually exclusive. None of these decompositions are used in Eringen's work, hence the constitutive theories in Eringen's work are of concern.

10) In microdilation theory additional one equation is needed for closure. Eringen proposes a new balance law to obtain this, balance of moments of symmetric stresses with gradients of the symmetric part of moment tensor. This law is not supported by classical thermodynamics, hence its use will yield thermodynamically inconsistent theory.

11) Eringen proposes conservation of microinertia to obtain three equations needed for closure. This conservation law is also not supported by classical thermodynamics, hence its use will lead to thermodynamically inconsistent theory.

We have presented plenty of evidence based on thermodynamics and well-established principles of mathematics that Eringen's microcontinuum theories are thermodynamically and mathematically inconsistent, hence are not valid microcontinuum theories.

10. Summary and Conclusions

A nonlinear microdilation microcontinuum theory has been presented in which mechanism of nonlinear elasticity is considered for the microconstituents, for the solid medium and for the interaction of the microconstituents with the solid medium. This microdilation theory considers the microconstituents, the medium and the interaction of the microconstituent with the medium all undergoing finite strain, finite deformation. In the following, we summarize the work presented in the paper and draw some conclusions.

1) We have presented derivation of conservation and balance laws for microdilation theory in which the kinematics of microconstituents, the solid medium and the interaction of the microconstituents with the solid medium is nonlinear. A check on the closure of the mathematical model consisting of conservation and balance laws and the constitutive theories reveals that additional four equations are needed for closure of the mathematical model. We have shown in this paper that one of these four equations can be extracted from the balance of angular momenta. Upon further examination we find that classical thermodynamics has no means of providing these three additional equations needed for closure. At this stage: a) we can conclude that perhaps consideration of nonlinear kinematics of

microconstituents is not physical within the thermodynamic framework considered. b) We can use conservation of microinertia to obtain the needed three equations but at the expense of a thermodynamically inconsistent microcontinuum theory. We have provided complete derivation of conservation and balance laws for nonlinear kinematics of microconstituents for those who wish to use the conservation of microinertia for closure. We do not advocate this due to thermodynamic inconsistency of the resulting theory. c) If we use linear kinematics for microconstituents then the resulting theory has closure without the need for any additional conservation or the balance law.

2) In the theory presented here, care is taken to ensure that the rigid body rotation physics of microconstituent that is common to all three microcontinuum theories is incorporated in identical manner in all three microcontinuum theories.

3) Our work recognizes that rotations ${}_c\Theta$ and Cauchy moment tensor is a new kinematically conjugate pair in all three microcontinuum theories, hence it requires two balance laws just as displacements and forces kinematic pair does in classical continuum mechanics. This necessitates new balance law in all microcontinuum theories [40] [50] [58], balance of moment of moments. This balance law is never used in Eringen's work, the consequence of this is spurious conjugate pairs in the entropy inequality, spurious constitutive theories and lack of thermodynamic equilibrium.

4) Varying rotations ${}_c\Theta$ in the deforming solid medium when resisted, create moments. Our derivation shows that the Cauchy moment tensor and the symmetric part of the gradients of ${}_c\Theta$ are kinematically work conjugate. This physics is purely due to nonclassical mechanics, hence has no interaction or any connection to classical continuum theory. Based on this, the 'integral-average' definition of moment tensor by Eringen's work is incorrect as it is based on $\bar{\sigma}^{(\alpha)}$ which is purely due to classical continuum mechanics.

5) Our derivation in this paper shows that the use of weight function $\bar{\phi}^{(\alpha)}(\bar{x}_m^{(\alpha)})$ in the derivation of macro balance of linear momenta, balance of angular momenta and moment of momentum has no thermodynamic basis.

6) In our work, all constitutive tensors of rank two are symmetric tensors and their argument tensors of rank two are also symmetric tensors, hence permitting the use of representation theorem in deriving constitutive theory that are naturally mathematically consistent. This is in contrast with published works in which the constitutive tensors of rank two are nonsymmetric tensors with nonsymmetric argument tensors. Such constitutive theories derived using assumed potentials are non physical and not justified based on representation theorem.

7) Conservation of microinertia advocated by Eringen to be necessary in microcontinuum theories is neither needed in the present work nor used. This conservation law is not supported by classical thermodynamics. The need for this law is primarily due to either ${}_a\Theta$ being rigid rotation of the microconstituents as unknown degrees of freedom or due to ${}_c\Theta^{(\alpha)}$ as degrees of freedom, whereas in our work ${}_a\Theta$ are in fact ${}_c\Theta$, hence are known and linear kinematics of

microconstituents does not require ${}_{\alpha}\Theta^{(\alpha)}$ as degrees of freedom. Other significant differences are that in Eringen's work σ and m are non symmetric and nine constitutive equations are considered for σ as well as m . In our work, $\sigma = {}_s\sigma + {}_a\sigma$ decomposition is used and there are only six constitutive equations needed for ${}_s\sigma$ as ${}_a\sigma$ is defined by balance of angular momenta. m is symmetric due to balance of moment of moments balance law, hence only six constitutive equations are needed for m as well. Eringen's microstretch theory does not have closure without conservation of microinertia conservation law primarily due to ${}_{\alpha}\Theta$.

8) Thermodynamic and mathematical consistency of the nonlinear microdilation theory with linear microconstituent kinematics presented in this paper has been established in section 8. The lack of thermodynamic and mathematical consistency of Eringen's linear microstretch theory has been discussed and illustrated in section 9.

9) It is established that with nonlinear microconstituent kinematics a thermodynamically and mathematically consistent microdilation microcontinuum theory is not possible as in this case classical thermodynamics has no means of providing three additional equations needed for closure of the mathematical model. This is perhaps an indication that nonlinear deformation of the microconstituents is not physical. We do not advocate using conservation of microinertia conservation law as this law proposed by Eringen has no thermodynamic basis, hence the resulting theory is not a valid theory. By assuming linear microconstituent kinematics but nonlinear deformation for the solid medium and for the interaction of the microconstituents with the solid medium, we have shown that the resulting microdilation theory is thermodynamically and mathematically consistent without the need of conservation and/or balance laws that are not supported by classical thermodynamics.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix A. Nonlinear Deformation of Microconstituents

In this case $\mathbf{S}^* : \dot{\mathbf{J}}^{(\alpha)}$ will appear as rate of work conjugate pair in the entropy inequality. Since microconstituent deformation is described by classical continuum mechanics, we have the following relationship.

$$\mathbf{S}^* : \dot{\mathbf{J}}^{(\alpha)} = \mathbf{S}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]}^{(\alpha)} \quad (\text{A.1})$$

In which $\mathbf{S}^{[0]}$ and $\dot{\boldsymbol{\varepsilon}}_{[0]}^{(\alpha)}$ are symmetric contravariant second Piola-Kirchhoff stress tensor due to microconstituents (volume average $\boldsymbol{\sigma}^{(\alpha)}$) or $\dot{\boldsymbol{\varepsilon}}^{(\alpha)}$ is the symmetric covariant nonlinear Green's strain tensor that is work conjugate to $\mathbf{S}^{[0]}$. The entropy inequality can be written as

$$\rho_0 \left(\dot{\Phi} + \eta \dot{\theta} \right) - {}^e_s \boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - {}^d_s \boldsymbol{\sigma}^{[0]} : \dot{\boldsymbol{\varepsilon}}_{[0]} - \mathbf{S} : \dot{\boldsymbol{\varepsilon}}^{(\alpha)} - \mathbf{m}^{[0]} : \left({}^c_{\ominus} \dot{\boldsymbol{\varepsilon}}_{[0]} \right) + \frac{\mathbf{q} \cdot \mathbf{g}}{\theta} \leq 0 \quad (\text{A.2})$$

Details of $\mathbf{S}^{[0]}$ and $\dot{\boldsymbol{\varepsilon}}_{[0]}^{(\alpha)}$ tensor are given in the following to ensure that their symmetric is not affected by purely volumetric deformation assumption followed by the consideration for the closure of the complete mathematical model.

Let $\mathbf{S}^{(0)}$ be Cauchy stress, then $\mathbf{S}^{[0]}$ is given by

$$\mathbf{S}^{[0]} = \left| \mathbf{J}^{(\alpha)} \right| \left(\mathbf{J}^{(\alpha)} \right)^{-1} \mathbf{S}^{(0)} \left(\left(\mathbf{J}^{(\alpha)} \right)^T \right)^{-1} \quad (\text{A.3})$$

In which $\mathbf{J}^{(\alpha)}$ is given by

$$\mathbf{J}^{(\alpha)} = {}^d_s \mathbf{J}^{(\alpha)} + {}^d_a \mathbf{J}^{(\alpha)} + \boldsymbol{\delta} = \varepsilon_v \boldsymbol{\delta} + {}^d_a \mathbf{J}^{(\alpha)} + \boldsymbol{\delta} \quad (\text{A.4})$$

or

$$\mathbf{J}^{(\alpha)} = \varepsilon_v \boldsymbol{\delta} + \boldsymbol{\delta} + {}^d_a \mathbf{J}^{(\alpha)} \quad (\text{A.5})$$

In which ${}^d_a \mathbf{J}^{(\alpha)}$ contains ${}^c_{\ominus} \boldsymbol{\Theta}^{(\alpha)}$, classical rotation in the volume $\bar{V}^{(\alpha)} + \partial \bar{V}^{(\alpha)}$ of the microconstituent and is given by

$$\left[{}^d_a \mathbf{J}^{(\alpha)} \right] = \frac{1}{2} \begin{bmatrix} 0 & -{}_c \boldsymbol{\Theta}_3^{(\alpha)} & {}_c \boldsymbol{\Theta}_2^{(\alpha)} \\ {}_c \boldsymbol{\Theta}_3^{(\alpha)} & 0 & -{}_c \boldsymbol{\Theta}_1^{(\alpha)} \\ -{}_c \boldsymbol{\Theta}_2^{(\alpha)} & {}_c \boldsymbol{\Theta}_1^{(\alpha)} & 0 \end{bmatrix} \quad (\text{A.6})$$

and $\mathbf{S}^{(0)}$ is pure pressure field define by

$$\mathbf{S}^{(0)} = s \boldsymbol{\delta} \quad (\text{A.7})$$

s is the pressure value.

Using (A.7) in (A.3)

$$\mathbf{S}^{[0]} = s \left| \mathbf{J}^{(\alpha)} \right| \left(\left(\mathbf{J}^{(\alpha)} \right)^T \cdot \left(\mathbf{J}^{(\alpha)} \right) \right)^{-1} \quad (\text{A.8})$$

Substituting (A.5) for $\mathbf{J}^{(\alpha)}$ in the inverse term.

$$\mathbf{S}^{[0]} = s \left| \mathbf{J}^{(\alpha)} \right| \left(\left(\varepsilon_v \boldsymbol{\delta} + \boldsymbol{\delta} + \left({}^d_a \mathbf{J}^{(\alpha)} \right)^T \right) \cdot \left(\varepsilon_v \boldsymbol{\delta} + \boldsymbol{\delta} + {}^d_a \mathbf{J}^{(\alpha)} \right) \right)^{-1} \quad (\text{A.9})$$

Simplifying (A.9) gives the following, still symmetric (as expected)

$$\mathbf{S}^{[0]} = s \left| \mathbf{J}^{(\alpha)} \right| \left(\varepsilon_v^2 \boldsymbol{\delta} + 2 \varepsilon_v \boldsymbol{\delta} + \boldsymbol{\delta} - {}^d_a \mathbf{J}^{(\alpha)} \cdot {}^d_a \mathbf{J}^{(\alpha)} \right) \quad (\text{A.10})$$

Consider symmetric Green's strain tensor $\boldsymbol{\varepsilon}_{[0]}^{(\alpha)}$

$$\boldsymbol{\varepsilon}_{[0]}^{(\alpha)} = \frac{1}{2} \left(\left(\mathbf{J}^{(\alpha)} \right)^T \cdot \mathbf{J}^{(\alpha)} - \mathbf{I} \right) \quad (\text{A.11})$$

Substituting $\mathbf{J}^{(\alpha)}$ from (A.5)

$$\boldsymbol{\varepsilon}_{[0]}^{(\alpha)} = \frac{1}{2} \left(\left(\varepsilon_v \boldsymbol{\delta} + \boldsymbol{\delta} + {}^d_a \mathbf{J}^{(\alpha)} \right)^T \cdot \left(\varepsilon_v \boldsymbol{\delta} + \boldsymbol{\delta} + {}^d_a \mathbf{J}^{(\alpha)} \right) - \boldsymbol{\delta} \right) \quad (\text{A.12})$$

Or

$$\boldsymbol{\varepsilon}_{[0]}^{(\alpha)} = \frac{1}{2} \left(\varepsilon_v^2 \boldsymbol{\delta} + 2\varepsilon_v \boldsymbol{\delta} - {}^d_a \mathbf{J}^{(\alpha)} \cdot {}^d_a \mathbf{J}^{(\alpha)} \right) \quad (\text{A.13})$$

In which ${}^d_a \mathbf{J}^{(\alpha)} \cdot {}^d_a \mathbf{J}^{(\alpha)}$ is given by

$$\begin{aligned} & \left[{}^d_a \mathbf{J}^{(\alpha)} \right] \left[{}^d_a \mathbf{J}^{(\alpha)} \right] \\ &= \frac{1}{2} \begin{bmatrix} \left({}_c \Theta_2^{(\alpha)} \right)^2 + \left({}_c \Theta_3^{(\alpha)} \right)^2 & {}_c \Theta_1^{(\alpha)} {}_c \Theta_2^{(\alpha)} & {}_c \Theta_1^{(\alpha)} {}_c \Theta_3^{(\alpha)} \\ {}_c \Theta_2^{(\alpha)} {}_c \Theta_1^{(\alpha)} & \left({}_c \Theta_1^{(\alpha)} \right)^2 + \left({}_c \Theta_3^{(\alpha)} \right)^2 & {}_c \Theta_2^{(\alpha)} {}_c \Theta_3^{(\alpha)} \\ {}_c \Theta_3^{(\alpha)} {}_c \Theta_1^{(\alpha)} & {}_c \Theta_3^{(\alpha)} {}_c \Theta_2^{(\alpha)} & \left({}_c \Theta_1^{(\alpha)} \right)^2 + \left({}_c \Theta_2^{(\alpha)} \right)^2 \end{bmatrix} \quad (\text{A.14}) \end{aligned}$$

With ${}^d_a \mathbf{J}^{(\alpha)} \cdot {}^d_a \mathbf{J}^{(\alpha)}$ given by (A.14), we note that $\mathbf{S}^{[0]}$ and $\boldsymbol{\varepsilon}_{[0]}^{(\alpha)}$ in (A.10) and (A.13) are symmetric (as expected) tensors of rank two. Consider work conjugate pair $\mathbf{S}^{[0]} : \boldsymbol{\varepsilon}_{[0]}^{(\alpha)}$ and substitute for $\boldsymbol{\varepsilon}_{[0]}^{(\alpha)}$ from (A.13) while keeping $\mathbf{S}^{[0]}$, a symmetric tensor of rank two with six independent components.

$$\mathbf{S}^{[0]} : \boldsymbol{\varepsilon}_{[0]}^{(\alpha)} = \mathbf{S}^{[0]} : \left(\frac{1}{2} \left(\varepsilon_v^2 \boldsymbol{\delta} + 2\varepsilon_v \boldsymbol{\delta} - {}^d_a \mathbf{J}^{(\alpha)} \cdot {}^d_a \mathbf{J}^{(\alpha)} \right) \right) \quad (\text{A.15})$$

In (A.10), $\mathbf{S}^{[0]}$ is the constitutive tensor and $\boldsymbol{\varepsilon}_{[0]}^{(\alpha)}$ is its argument tensor. Thus, for thermoelastic nonlinear microconstituent deformation we can write

$$\mathbf{S}^{[0]} = \left(\boldsymbol{\varepsilon}_{[0]}^{(\alpha)}, \theta \right) \quad (\text{A.16})$$

Equation (A.16) provides six constitutive equations and six components of $\boldsymbol{\varepsilon}_{[0]}^{(\alpha)}$ that are functions of four degrees of freedom ε_v and ${}_c \Theta^{(\alpha)}$. Thus we need four additional equations for closure.

1) Balance of angular momenta provides one equation, Equation (159) obtained by premultiplying balance of angular momenta with the inverse of the permutation function.

2) We need additional three equations for closure of the mathematical model.

3) As explained in 6.8.3, classical thermodynamics provides no means of deriving or extracting these from the existing balance laws.

4) An alternative is to use linear micro deformation, then the resulting model has closure.

5) As suggested by Eringen if we use conservation of microinertia conservation law, we obtained three additional equations needed for closure. In this case the material presented in this appendix applies and (A.16) can be used for deriving constitutive theories. The major problem in this approach is that conservation of

microinertia is not a conservation law in classical thermodynamics, hence its use will result in a thermodynamically inconsistent microcontinuum theory.

6) The published works that use this conservation and balance law can indeed use (A.16) for deriving constitutive theory for nonlinear thermoelastic case as well as its extensions to nonlinear dissipation and rheology physics for microconstituents.

7) We note that for linear deformation of microconstituents $\boldsymbol{\varepsilon}_{[0]}^{(\alpha)}$ in (A.13) reduces to $\boldsymbol{\varepsilon}^{(\alpha)} = \varepsilon_v \boldsymbol{\delta}$ as expected.