Scientific Research Publishing

Applied Mathematics, 2026, 17(5), 281-296
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Spectral Extremal Graphs for the F_3 Graph

Yizhi Yin

School of Mathematical Sciences, Shenzhen University, Shenzhen, China
Email: 838474243@qq.com

How to cite this paper: Yin, Y.Z. (2026) Spectral Extremal Graphs for the F_3 Graph. *Applied Mathematics*, **17**, 281-296.

https://doi.org/10.4236/am.2026.175017

Received: January 15, 2026
Accepted: May 26, 2026
Published: May 29, 2026

Abstract

Let F_k be the (friendship) graph obtained from k triangles by sharing a common vertex. In 2024, Li, Lu, Peng [Discrete Mathematics 346(2023)] show that the unique n-vertex F_2-free spectral extremal graph is the balanced bipartite graph adding an edge in smaller part if $n \geqslant 7$. Following their result, we show that the unique n-vertex F_3-free spectral extremal graph is the balanced complete bipartite graph adding two disjoint K_3 in the vertex part with smaller size if $n > 360$.

Keywords

F_3-Free, Spectral Extremal Graph, Balanced Complete Bipartite Graph

1. Introduction

In this paper, we shall use the following standard notation (see [1]) and consider only simple and undirected graphs. Let G be a graph with vertex set $V(G) = \{v_1, ..., v_n\}$ and edge set $E(G) = \{e_1, ..., e_m\}$, $A(G)$ be the adjacency matrix of G. Let $\lambda(G)$ be the spectral radius of G, which is defined as the maximum modulus of eigenvalues of $A(G)$. Denote the matching number of G by $\alpha'(G)$. A graph G is called F-free if it does not contain an isomorphic copy of F as a subgraph. The *Turán* number of a graph F is the maximum number of edges in an n-vertex F-free graph, denoted by $ex(n, F)$, that is,

$$ex(n, F) := \max \{e(G) : |G| = n \text{ and } F \nsubseteq G\}.$$

An F-free graph on n vertices with $ex(n, F)$ edges is called an extremal graph for F. We write $Ex(n, F)$ for the set of all n-vertex F-free graphs with maximum number of edges.

Let F_k denote the k-fan graph, which is the graph consisting of k triangles that intersect in exactly one common vertex. In 1995, Erdős, Fredi, Gould and Gunderson [2] proved the following result.

Theorem 1.1. *(Erdős-Fredi-Gould-Gunderson [2], 1995). For every*

$k \geqslant 1$ *and* $n \geqslant 50k^2$,

$$ex(n, F_k) = \lfloor \frac{n^2}{4} \rfloor + \begin{cases} k^2 - k, & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k, & \text{if } k \text{ is even.} \end{cases}$$

The extremal graphs in $Ex(n, F_k)$ are also determined in [2] as follows.

(a) For odd k, the extremal graphs are constructed from $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ by embedding two vertex-disjoint copies of the complete graph K_k in one side.

(b) For even k, the extremal graphs are obtained by taking $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ and embedding a graph with $2k-1$ vertices, $k^2 - \frac{3}{2}k$ edges and maximum degree $k-1$ in one side.

For fixed $k \geqslant 2$ and sufficiently large order n, the spectral extremal problem for F_k was recently characterized by Cioabă, Feng, Tait and Zhang [3].

Theorem 1.2. *(Cioabă-Feng-Tait-Zhang [3], 2020). Let $k \geqslant 2$ and G be an F_k-free graph on n vertices. For sufficiently large n, if G has the maximal spectral radius, then*

$$G \in Ex(n, F_k).$$

In 2022, Zhai, Liu and Xue [4] provided a further characterization of G and determined the unique spectral extremal graph.

Denoted by $K^-_{a,b}$ the graph obtained from the balanced complete bipartite graph $K_{a,b}$ by adding an edge in the smaller part. In 2023, Li, Lu and Peng give a spectral extremal result of F_2-free graph without the condition on n being sufficiently large.

Theorem 1.3. *(Li-Lu-Peng [5]). If $n \geqslant 7$ and G is an F_2-free graph on n vertices, then*

$$\lambda(G) \leqslant \lambda(K^-_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}),$$

equality holds if and only if $G = K^-_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

Recently, many scholars are interested in spectral extremal graphs for F_k-free graphs with given size. The spectral extremal problems for F_k-free graphs with given size have been solved when $k \geqslant 2$ (see [6-11]). It's also intersecting to consider the spectral extremal problems for F_k-free graphs with given order. Li, Feng and Peng [11] have proved that the spectral extremal result [[3], Theorem 2] for F_k-free graphs holds for the every $n \geqslant (21k)^4$. They asked whether the spectral extremal result for F_k still hold for every $n \geqslant Ck$ with an absolute constant $C > 0$.

Denoted by $K^*_{a,b}$ the graph obtained from the balanced complete bipartite graph $K_{a,b}$ by embedding two disjoint K_3 in the smaller part. Our main result is as follows.

Theorem 1.4. *If $n > 360$ and G is an F_3-free graph on n vertices, then*

$$\lambda(G) \leqslant \lambda(K^*_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}),$$

equality holds if and only if $G = K^*_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

This paper is organized as follows. In Section 2, we present some preliminary lemmas. In Section 3, the proof of Theorem 1.4 will be provided.

2. Preliminaries

Theorem 2.1. *(Pigeonhole Principle [1]). If $n+1$ letters are distributed among n pigeonholes, at least two of them will end up in the same pigeonhole.*

Lemma 2.2. *[12] If G_2 is a proper subgraph of a graph G_1, then*

$$\lambda(G_1) > \lambda(G_2).$$

The Kelmans transformation S_{xy} for graph G is defined: $S_{xy}(G) := \{S_{xy}(e) : e \in E(G)\}$, where

$$S_{xy}(e) = \begin{cases} (e\setminus\{y\}) \cup \{x\}, & \text{if } y \in e,\ x \notin e \text{ and } (e\setminus\{y\}) \cup \{x\} \notin E(G); \\ e, & \text{otherwise.} \end{cases}$$

The following lemma is useful to compare spectral radius of similiar graphs.

Lemma 2.3. *(Csikvári [13]). Let u, v be two vertices of G, then*

$$\lambda(S_{uv}(G)) \geqslant \lambda(G).$$

Define $f(\alpha', \Delta) = \max\{|E(G)| : \alpha'(G) \leqslant \alpha', \Delta(G) \leqslant \Delta\}$. In 1976, Chvátal and Hanson give an upper bound of $e(G)$ with fixed $\Delta(G)$ and $\alpha'(G)$.

Lemma 2.4. *(V. Chvátal, D. Hanson [14]). If $\alpha'(G) \geqslant 1$ and $\Delta \geqslant 1$, then*

$$f(\alpha', \Delta) = \alpha'\Delta + \lfloor \frac{\Delta}{2} \rfloor \lfloor \frac{\alpha'}{\lceil \frac{\Delta}{2} \rceil} \rfloor \leqslant \alpha'\Delta + \alpha'.$$

Denoted by $K^*_{a,b}$ the graph obtained from the complete bipartite graph $K_{a,b}$ by adding two disjoint K_3 in the smaller part. Denoted by $K^p_{a,b}$ the graph obtained from the complete bipartite graph $K_{a,b}$ by embedding a C_p in the smaller part. The following lemma give the characterization of the spectral radius of $K^p_{a,b}$ and $K^*_{a,b}$, and the proof is inspired by Li, Lu, Peng [[5], Lemma 2.1].

Lemma 2.5. *$\lambda(K^p_{a,b})$ is the largest root of $f(x) = x^3 - 2x^2 - abx + 2ab - 2pb$ and $\lambda(K^*_{a,b})$ is the largest root of $f(x) = x^3 - 2x^2 - abx + 2ab - 12b$.*

Denoted by $K^{t_1}_{a,b}$ the graph obtained from the complete bipartite graph $K_{a,b}$ by adding a disjoint union of K_3 and P_4 in the smaller part. Denoted by $K^{t_2}_{a,b}$ the graph obtained from the complete bipartite graph $K_{a,b}$ by adding a P_7 in the smaller part. Denoted by $K^{t_3}_{a,b}$ the graph obtained from the complete bipartite graph $K_{a,b}$ by adding a disjoint union of a K_3 and a P_4 in the smaller part.

When $3 \leqslant p \leqslant 6$ from Lemma 2.5, we have $\lambda(K^p_{a,b}) \leqslant \lambda(K^6_{a,b}) = \lambda(K^*_{a,b}) < \lambda(K^*_{\lfloor n/2 \rfloor, \lceil n/2 \rceil})$. By Lemma 2.3, we know $\lambda(K^{t_1}_{a,b}), \lambda(K^{t_2}_{a,b}), \lambda(K^{t_3}_{a,b}) \leqslant \lambda(K^*_{a,b})$. From above we have

$$\lambda(K^p_{a,b}), \lambda(K^{t_1}_{a,b}), \lambda(K^{t_2}_{a,b}), \lambda(K^{t_3}_{a,b}) \leqslant \lambda(K^*_{a,b}) < \lambda(K^*_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}) \quad (1)$$

where $3 \leqslant p \leqslant 6$.

The following lemma is inspired by [3], it's useful to calculate component of the Perron vector in bipartite graph.

Lemma 2.6. *If G is a graph with vertex set $V(G)$, we divide $V(G)$ into two partition denoted by X and Y arbitrarily. Let $x = (x_1, \ldots, x_n)^T$ be a Perron vector of G, normalize x so $x_u = \max\{x_v : v \in G\} = 1$. If $u, v \in X$ and $d_X(u) = k$, then $x_v \geqslant 1 - \frac{\sum_{i\sim u, i\in Y, i\nsim v} 1 + k}{\lambda}$.*

Proof. Since $u, v \in X$ and $x_v \leqslant x_u = 1$ we have

$$\begin{aligned}
\lambda x_v - \lambda x_u = & \sum_{i\sim v, i\in Y, i\sim u} x_i + \sum_{i\sim v, i\in Y, i\nsim u} x_i + \sum_{i\sim v, i\in X} x_i \\
& - \sum_{i\sim u, i\in Y, i\sim v} x_i - \sum_{i\sim u, i\in Y, i\nsim v} x_i - \sum_{i\sim u, i\in X} x_i \\
> & \sum_{i\sim v, i\in Y, i\sim u} x_i + 0 + 0 - \sum_{i\sim u, i\in Y, i\sim v} x_i \\
& - \sum_{i\sim u, i\in Y, i\nsim v} x_i - \sum_{i\sim u, i\in X} x_i \\
= & - \sum_{i\sim u, i\in Y, i\nsim v} x_i - \sum_{i\sim u, i\in X} x_i \\
\geqslant & - \sum_{i\sim u, i\in Y, i\nsim v} 1 - k,
\end{aligned}$$

which leads to $x_v \geqslant 1 - \frac{\sum_{i\sim u, i\in Y, i\nsim v} 1 + k}{\lambda}$. □

3. Proof of the Main Result

Proof of Theorem 1.4. Let G be an F_3-free graph with maximum spectral radius, $x = (x_1, x_2, \ldots, x_n)^T$ be a Perron vector of G and $u \in V(G)$ be a vertex such that $x_u = \max\{x_v : v \in V(G)\}$. For notational convenience, we denote $\lambda = \lambda(G)$, $A = N_G(u)$, $B = V(G)\backslash(A \cup \{u\})$, $A^+ = \{v \in A \mid d_A(v) \geqslant 1\}$, $B^+ = \{v \in B \mid d_B(v) \geqslant 1\}$. Then $|A| + |B| + 1 = n$.

The following claim, in which we establish several bounds for certain parameters of graph G, is inspired by [2].

Claim 3.1. *The following conclusion holds.*

(a) $\lambda^2(G) \geqslant \lambda^2(K^*_{\lfloor n/2\rfloor, \lceil n/2\rceil}) > \lfloor \frac{n^2}{4} \rfloor + 12$.

(b) $|A| \geqslant \lambda$. *Furthermore, we have* $\lfloor \frac{n}{2} \rfloor < |A| < \frac{n+4}{2} + \sqrt{2n-8}$, $\frac{n-6}{2} - \sqrt{2n-8} < |B| < \lfloor \frac{n}{2} \rfloor - 1$. *When* $n > 360$, *we have* $\lceil \frac{n}{2} \rceil \leqslant |A| < \frac{7}{12}n$, $\frac{5}{12}n < |B| \leqslant \lfloor \frac{n}{2} \rfloor - 1$.

(c) $e(A) \geqslant 7$.

(d) $e(A, B) + 2e(A) > |A|\,|B| - 2\,|B| + 2\,|A| + 6$.

(e) *G is connected.*

Proof. (a) Since G is an F_3-free graph with maximum spectral radius, $\lambda(G) \geqslant \lambda(K^*_{\lfloor n/2\rfloor, \lceil n/2\rceil})$. It suffices to prove $\lambda^2(K^*_{\lfloor n/2\rfloor, \lceil n/2\rceil}) > \lfloor \frac{n^2}{4} \rfloor + 12$. For even n, note that $\lambda^2(K^*_{\lfloor n/2\rfloor, \lceil n/2\rceil}) > (\frac{2e(K^*_{\lfloor n/2\rfloor, \lceil n/2\rceil})}{n})^2 = \frac{n^2}{4} + 12 + \frac{144}{n^2} > \lfloor \frac{n^2}{4} \rfloor + 12$.

For odd n, we construct a unit vector $x := (\frac{1}{\sqrt{n-1}}, \ldots, \frac{1}{\sqrt{n-1}}, \frac{1}{\sqrt{n+1}}, \ldots, \frac{1}{\sqrt{n+1}})^T$, where $\frac{1}{\sqrt{n-1}}$ corresponds to the vertex of smaller part and $\frac{1}{\sqrt{n+1}}$ corresponds to the larger part. Then Rayleigh's formula implies $\lambda(K^*_{\lfloor n/2\rfloor, \lceil n/2\rceil}) >$

$x^T A(K^*_{\lfloor n/2\rfloor,\lceil n/2\rceil})x > \frac{\sqrt{n^2-1}}{2} + \frac{12}{n-1}$, which leads to $\lambda^2(K^*_{\lfloor n/2\rfloor,\lceil n/2\rceil}) > \frac{n^2-1}{4} + 12 = \lfloor \frac{n^2}{4} \rfloor + 12$. Hence we have

$$\lambda^2(G) \geqslant \lambda^2(K^*_{\lfloor n/2\rfloor,\lceil n/2\rceil}) > \lfloor \frac{n^2}{4} \rfloor + 12. \tag{2}$$

(b) By the maximality of x_u, we get $\lambda x_u = \sum_{v\in A} x_v \leqslant |A|\, x_u$. Hence $\lambda \leqslant |A|$. From inequality (2), we get $\lambda > \frac{n}{2}$. Since $|A|$ is an integer, we have $|A| \geqslant \lceil \frac{n}{2} \rceil$. Keeping in mind that $|A| + |B| = n - 1$, we obtain $|B| \leqslant \lfloor \frac{n}{2} \rfloor - 1$.

Denoted by $d_A(v)$ the number of neighbors of v in vertex set A. Namely, $d_A(v) = |N_G(v) \cap A|$. It is known that

$$\lambda^2 x_u = \sum_{v\in A} \sum_{w\in N(v)} x_w = |A|\, x_u + \sum_{v\in A} d_A(v) x_v + \sum_{w\in B} d_A(w) x_w.$$

Since $\sum\limits_{v\in A} d_A(v) = 2e(A)$ and $\sum_{w\in B} d_A(w) = e(A, B)$, we have

$$\lambda^2 \leqslant e(A, B) + 2e(A) + |A|\,. \tag{3}$$

Since G is F_3-free, we have $\alpha'(G[A]) \leqslant 2$. From Lemma 2.4 and $\Delta(G[A]) \leqslant |A| - 1$, we have $e(A) \leqslant 2\,|A|$. Observe that $e(A, B) \leqslant |A|\,|B|$ and $e(A) \leqslant 2\,|A|$. Then we have $\lambda^2 \leqslant e(A, B) + 2e(A) + |A| \leqslant |A|\,|B| + 5\,|A|$. Since $\lambda^2(G) \geqslant \lambda^2(K^*_{\lfloor \frac{n}{2}\rfloor,\lceil \frac{n}{2}\rceil}) > \lfloor \frac{n^2}{4} \rfloor + 12$, we have $|A|\,|B| + 5\,|A| > \lfloor \frac{n^2}{4} \rfloor + 12$, then we get

$$\frac{n+4}{2} - \sqrt{2n-8} < |A| < \frac{n+4}{2} + \sqrt{2n-8},$$
$$\frac{n-6}{2} - \sqrt{2n-8} < |B| < \frac{n-6}{2} + \sqrt{2n-8}.$$

Since $n > 360$, $\sqrt{2n-8} + 3 < \frac{1}{12}n$. Then combined with $|A| \geqslant \lceil \frac{n}{2} \rceil$ and $|B| \leqslant \lfloor \frac{n}{2} \rfloor - 1$, we obtain

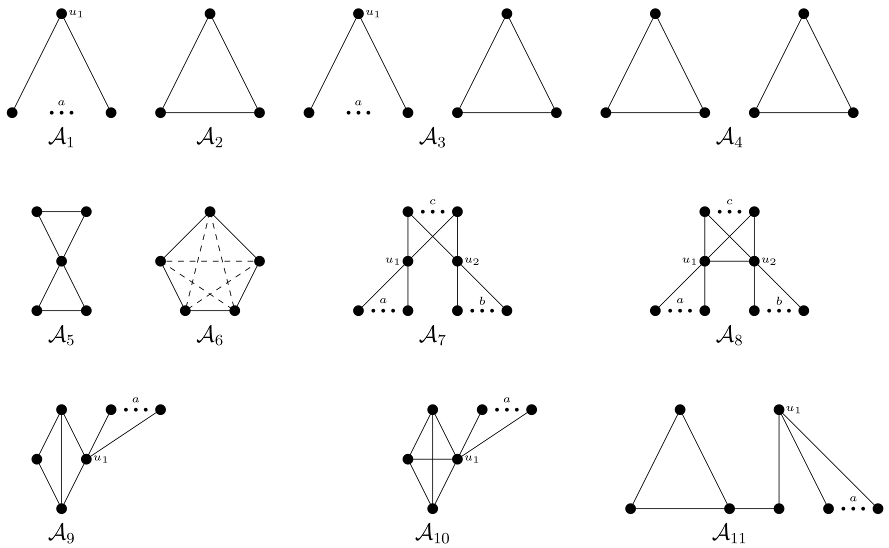
$$\lfloor \frac{n}{2} \rfloor < |A| < \frac{n}{2} + \frac{n}{12} = \frac{5}{12}n,$$
$$\frac{n}{2} - \frac{n}{12} = \frac{5}{12}n < |B| < \lfloor \frac{n}{2} \rfloor - 1.$$

(c) Suppose on the contrary that $e(A) \leqslant 6$. From (3), we have $\lambda^2 \leqslant e(A, B) + 2e(A) + |A|$. Note that $|A| + |B| + 1 = n$ and $|B| \leqslant \frac{n}{2} - 1$, we have $\lambda^2 \leqslant \lfloor \frac{n^2}{4} \rfloor + 12$, which contradicts with (2). Thus, we have proved $e(A) \geqslant 7$.

(d) From (3), we get $\lambda^2 \leqslant e(A, B) + 2e(A) + |A|$. Suppose on the contrary $e(A, B) + 2e(A) \leqslant |A|\,|B| - 2\,|B| + 2\,|A| + 6$, then we have $\lambda^2 \leqslant e(A, B) + 2e(A) + |A| \leqslant |A|\,|B| - 2\,|B| + 3\,|A| + 6 \leqslant \lfloor \frac{n^2}{4} \rfloor + 12$, which contradics with (2). Hence we have

$$e(A, B) + 2e(A) > |A|\,|B| - 2\,|B| + 2\,|A| + 6. \tag{4}$$

(e) Suppose on the contrary, choose G_1 and G_2 as two components with $\lambda(G_1) = \lambda(G)$, and add an edge between G_1 and G_2, we get a new graph on n vertex, which is still F_3-free and has larger spectral radius than G, contradiction. □



Since G is F_3-free, we have $\alpha'(G[A^+]) \leqslant 2$ and $G[A^+]$ is P_6-free. Hence we can divide all possible structure of $G[A^+]$ into $\{\mathcal{A}_1, \ldots, \mathcal{A}_{11}\}$.

Claim 3.2. $\bigcup_{i=1}^{11} \mathcal{A}_i$ *are all possible structure of* $G[A^+]$.

Proof. Firstly we consider when $G[A^+]$ is connected.

Case 1. The circumference of $G[A^+]$ is 5.

Denote the vertex of C_5 in $G[A^+]$ by v_1, v_2, v_3, v_4, v_5 in order. Since $\alpha'(G[A^+]) \leqslant 2$, there exists no edges between $\{v_1, v_2, v_3, v_4, v_5\}$ and $A - v_1 - v_2 - v_3 - v_4 - v_5$. There may exist edges in $\{v_1, v_2, v_3, v_4, v_5\}$. See $\mathcal{A}_6$.

Case 2. The circumference of $G[A^+]$ is 4.

Denote the vertex of C_4 in $G[A^+]$ by v_1, v_2, v_3, v_4 in order.

Subcase 2.1. $v_1 \nsim v_3$, $v_2 \nsim v_4$.

Since $\alpha'(G[A^+]) \leqslant 2$, there exists no $2K_2$ between $\{v_1, v_2\}$ (or $\{v_2, v_3\}$ or $\{v_3, v_4\}$ or $\{v_1, v_4\}$) and $A - v_1 - v_2 - v_3 - v_4$. There may exist $2K_2$ between $\{v_1, v_3\}$ (or $\{v_2, v_4\}$) and $A - v_1 - v_2 - v_3 - v_4$. See $\mathcal{A}_7$ with $a \geqslant 0, b \geqslant 0, c \geqslant 2$.

Subcase 2.2. $v_1 \nsim v_3$, $v_2 \sim v_4$.

Since $\alpha'(G[A^+]) \leqslant 2$, there exists no $2K_2$ between $\{v_1, v_3\}$ (or $\{v_1, v_2\}$ or $\{v_2, v_3\}$ or $\{v_3, v_4\}$ or $\{v_1, v_4\}$) and $A - v_1 - v_2 - v_3 - v_4$. There may exist $2K_2$ between $\{v_2, v_4\}$ and $A - v_1 - v_2 - v_3 - v_4$. See $\mathcal{A}_8$ with $a \geqslant 0, b \geqslant 0, c \geqslant 2$. There may exist K_2 between $\{v_1, v_2, v_3, v_4\}$ and $A - v_1 - v_2 - v_3 - v_4$. See $\mathcal{A}_8$ with $a \geqslant 0, b \geqslant 0, c \geqslant 2$ or $\mathcal{A}_9$ with $a \geqslant 0$.

Subcase 2.3. $v_1 \sim v_3$, $v_2 \sim v_4$.

Since $\alpha'(G[A^+]) \leqslant 2$, there exists no $2K_2$ between $\{v_1, v_3\}$ (or $\{v_1, v_2\}$ or $\{v_2, v_3\}$ or $\{v_3, v_4\}$ or $\{v_1, v_4\}$ or $\{v_2, v_4\}$) and $A - v_1 - v_2 - v_3 - v_4$. There may exist K_2 between $\{v_1, v_2, v_3, v_4\}$ and $A - v_1 - v_2 - v_3 - v_4$. See $\mathcal{A}_{11}$ with $a \geqslant 0$.

Case 3. The circumference of $G[A^+]$ is 3.

Denote the vertex of the C_3 in $G[A^+]$ by v_1, v_2, v_3 in order. Since $\alpha'(G[A^+]) \leqslant 2$, there exists no $3K_2$ between $\{v_1, v_2, v_3\}$ and $A - v_1 - v_2 - v_3$.

Subcase 3.1. There exists a $2K_2$ between $\{v_1, v_2\}$ and $A - v_1 - v_2 - v_3$.

See $\mathcal{A}_8$ with $a \geqslant 0, b \geqslant 0, c \geqslant 1$.

Subcase 3.2. There exists a K_2 between $\{v_1\}$ and $A - v_1 - v_2 - v_3$.

Suppose $v_4 \in A - v_1 - v_2 - v_3$ and $v_1 \sim v_4$. Since $\alpha'(G[A^+]) \leqslant 2$, there exists no $2K_2$ between $\{v_1, v_4\}$ and $A - v_1 - v_2 - v_3 - v_4$. There may exist K_2 between $\{v_4\}$ and $A - v_1 - v_2 - v_3 - v_4$. See $\mathcal{A}_{11}$ with $a \geqslant 1$. There may exist K_2 between $\{v_1\}$ and $A - v_1 - v_2 - v_3 - v_4$. See $\mathcal{A}_8$ with $a \geqslant 0, b \geqslant 0, c \geqslant 1$. There may exist K_2 between $\{v_1, v_4\}$ and $A - v_1 - v_2 - v_3 - v_4$. See $\mathcal{A}_5$. There may exist no edges between $\{v_1, v_2, v_3, v_4\}$ and $A - v_1 - v_2 - v_3 - v_4$. See $\mathcal{A}_{11}$ with $a = 0$.

Case 4. There exists no cycle in $G[A^+]$.

Subcase 4.1. The length of the longest path in $G[A^+]$ is 5.

Denote the vertex of the P_5 in $G[A^+]$ by v_1, v_2, v_3, v_4, v_5 in order. Since $\alpha'(G[A^+]) \leqslant 2$, there exists no K_2 between $\{v_1, v_3, v_5\}$ and $A - v_1 - v_2 - v_3 - v_4 - v_5$. There may exist $2K_2$ between $\{v_2, v_4\}$ and $A - v_1 - v_2 - v_3 - v_4 - v_5$. See $\mathcal{A}_7$ with $a \geqslant 1, b \geqslant 1, c = 1$.

Subcase 4.2. The length of the longest path in $G[A^+]$ is 4.

Denote the vertex of the P_4 in $G[A^+]$ by v_1, v_2, v_3, v_4 in order. There may exist $2K_2$ between $\{v_2, v_3\}$ and $A - v_1 - v_2 - v_3 - v_4$. See $\mathcal{A}_8$ with $a \geqslant 1, b \geqslant 1, c \geqslant 0$.

Subcase 4.3. The length of the longest path in $G[A^+]$ is 3.

Denote the vertex of the P_3 in $G[A^+]$ by v_1, v_2, v_3 in order. There may exist K_2 between $\{v_2\}$ and $A - v_1 - v_2 - v_3$. See $\mathcal{A}_1$ with $a \geqslant 2$.

Subcase 4.4. The length of the longest path in $G[A^+]$ is 2.

See $\mathcal{A}_1$ with $a = 1$.

Secondly we consider when $G[A^+]$ is disconnected. Since $\alpha'(G[A^+]) = 2$, the circumference of $G[A^+]$ is at most 3 and $G[A^+]$ is P_4-free. Then every component of $G[A^+]$ is a K_3 or a star. See $\mathcal{A}_3$ with $a \geqslant 3$, $\mathcal{A}_4$ and $\mathcal{A}_7$ with $a \geqslant 0, b \geqslant 0, c = 0$. □

By calculation on $e(A, B)$ and $\lambda(G)$, when $G[A^+] \in \bigcup_{i \in \{1,2,3,4,5,6,9,10,11\}} \mathcal{A}_i$ we can prove a contradiction.

Claim 3.3. *If* $n > 360$*, then* $G[A^+] \notin \bigcup_{i \in \{1,2,3,4,5,6,9,10,11\}} \mathcal{A}_i$.

Proof. When $G[A^+] \in \bigcup_{i \in \{1,3,9,10,11\}} \mathcal{A}_i$. Denote the neighborhoods of u_1 in $G[A^+]$ by $\{v_1, \ldots, v_t\}$. Let $X = B \cup \{u, u_1\}$, $X^+ = \{v \in X \mid d_X(v) \geqslant 1\}$, $Y = A - u_1$, $Y^+ = \{v \in Y \mid d_Y(v) \geqslant 1\}$. Easily note that $t \leqslant |A| - 1$.

Case 1. $G[A^+] \in \mathcal{A}_1$. Firstly, we claim $|N_B(u_1)| \leqslant 1$, otherwise suppose on the contrary $|N_B(u_1)| \geqslant 2$. From Claim 3.1, we know $t \geqslant 7$. Avoid F_3 at u_1, by Theorem 2.1 we have $e(N_A(u_1), N_B(u_1)) \leqslant \max\{|N_B(u_1)|, t\}$, then $e(A, B) + 2e(A) \leqslant |A||B| - |B| + |A| + 1$, which leads a contradiction with (4), hence $|N_B(u_1)| \leqslant 1$.

Secondly, we claim $G[X]$ is $K_{1,3}$-free and $3K_2$-free. On the contrary, suppose $G[X]$ contain a $K_{1,3}$, denote the center vertex of the $K_{1,3}$ by c_1, three vertex in $N_X(c_1)$ by c_2, c_3, c_4. Since $N_B(u_1) \leqslant 1$ and $N_X(c_1) \geqslant 3$, $c_1 \in B = X - u - u_1$.

When $|N_Y(c_1)| \geqslant 3$. Avoid F_3 at c_1, there exists $i \in \{2, 3, 4\}$, satisfy $e(c_i, N_Y(c_1)) \leqslant 2$ and $e(c_1, Y) + e(c_i, Y) \leqslant |Y| + 2$.

When $|N_Y(c_1)| \leqslant 2$. For $i \in \{2, 3, 4\}$, we have $e(c_i, N_Y(c_1)) \leqslant 2$ and $e(c_1, Y) + e(c_i, Y) \leqslant |Y| + 2$.

Let c_2 be a vertex in $N_X(c_1)$, satisfy $e(c_1, Y) + e(c_2, Y) \leqslant |Y| + 2$.

If $c_2 \in B$, we have $e(A, B) + 2e(A) \leqslant |A||B| - |B| + |A| + 1 + |N_B(u_1)|$, which leads a contradiction with (4). If $c_2 = u_1$, we have $e(A, B) + 2e(A) \leqslant |A||B| - |B| + |A| + |N_B(u_1)| + 1$, which leads a contradiction with (4). Hence $G[X]$ is $K_{1,3}$-free.

When $G[X]$ contain a $3K_2$, denote 3 disjoint edges in $G[X]$ by c_1c_2, c_3c_4, c_5c_6. Avoid F_3 at $A - u_1$, we have $e(\{c_1, \ldots, c_6\}, Y) \leqslant 5|Y|$.

If $c_1, c_2, c_3, c_4, c_5, c_6 \in B$, then we have $e(A,B) + 2e(A) \leqslant |A|\,|B| + |A| - |B| + |N_B(u_1)| - 1$, which leads a contradiction with (4). If $c_1 = u_1$, then we have $e(A,B) + 2e(A) \leqslant |A|\,|B| + |A| - |B| + |N_B(u_1)| - 1$, which leads a contradiction with (4). Hence $G[X]$ is $3K_2$-free.

Since $G[X]$ is $K_{1,3}$-free and $3K_2$-free, we know $G[B^+] \in \{K_{1,2}, K_2, K_3, \varnothing\}$. Combined with $N_B(u_1) \leqslant 1$ we note that $G[X^+]$ is a proper subgraph of two disjoint K_3 or a P_7. And $G[Y^+]$ is empty. Hence G is a proper subgraph of $K^*_{a,b}$ or $K^{t_2}_{a,b}$, which leads to $\lambda(G) < \lambda(K^*_{\lfloor n/2\rfloor,\lceil n/2\rceil})$, contradiction.

Case 2. $G[A^+] \in \mathcal{A}_3$. Denote the vertex of K_3 in $G[A^+]$ by t_1, t_2, t_3. Similiar with case 1, we can prove $|N_B(u_1)| \leqslant 1$ and $G[X]$ is $K_{1,3}$-free and $3K_2$-free, hence $G[B^+] \in \{K_{1,2}, K_2, K_3, \varnothing\}$.

We complete the edges between X and $Y - t_1 - t_2 - t_3$ and the edges between $X - u_1$ and $t_1 + t_2 + t_3$ in graph G denote the result graph by G' (G' is not F_3-free anymore). Since G is a subgraph of G', we have $\lambda(G) \leqslant \lambda(G')$. Note that $\lambda(G') < \frac{7}{12}n + \frac{1}{360}n$. Let $y = (y_1, \ldots, y_n)^T$ be a Perron vector of G', normalize y so that $y_{u'} = \max\{y_v : v \in G'\} = 1$.

Subcase 2.1. $u' \in X$.

For $v \in X$ in graph G', we have $\{i \in Y : i \sim u', i \nsim v\} \subset \{t_1, t_2, t_3\}$ and $d_X(u') \leqslant \Delta(G[X]) = 2$. By Lemma 2.6 we have $y_v \geqslant 1 - \frac{5}{\lambda(G')} > 1 - \frac{10}{n} > \frac{35}{36}$ for $v \in X$.

For $v \in Y$, we have $y_v > \frac{69}{100}$. Hence $y_{u_1}y_{t_1} + y_{u_1}y_{t_2} + y_{u_1}y_{t_3} + y_u y_{b_1} > 3 \cdot \frac{35}{36} \cdot \frac{69}{100} + (\frac{35}{36})^2 > 3$.

Subcase 2.2. $u' \in Y$.

For $v \in Y$ in graph G', we have $\{i \in X : i \sim u', i \nsim v\} \subset \{u_1\}$ and $d_Y(u') \leqslant \Delta(G[Y]) = 2$. By Lemma 2.6 we have $y_v \geqslant 1 - \frac{3}{\lambda(G')} > 1 - \frac{6}{n} > \frac{59}{60}$ for $v \in Y$.

For $v \in X$, we have $y_v > \frac{41}{50}$. Hence $y_{u_1}y_{t_1} + y_{u_1}y_{t_2} + y_{u_1}y_{t_3} + y_u y_{b_1} > 3 \cdot \frac{41}{50} \cdot \frac{59}{60} + (\frac{41}{50})^2 > 3$.

Since $\Delta(G[X]) \leqslant 2$ and $\alpha'(G[X]) \leqslant 2$, combined with Lemma 2.4 we have $e(B) \leqslant e(X) \leqslant 6$. Since $|B| > \frac{5}{12}n > 2e(B)$, there exists an independent vertex b_1 in $G[B]$. Let $G'' = G' - t_1t_2 - t_2t_3 - t_1t_3 + u_1t_1 + u_1t_2 + u_1t_3 + ub_1$, we have

$$\begin{aligned}
\lambda(G'') - \lambda(G') =& y^T(A(G'') - A(G'))y \\
=& -(y_{t_1}y_{t_2} + y_{t_2}y_{t_3} + y_{t_1}y_{t_3}) + y_{u_1}y_{t_1} + y_{u_1}y_{t_2} \\
& + y_{u_1}y_{t_3} + y_u y_{b_1} \\
>& -3 + + y_{u_1}y_{t_1} + y_{u_1}y_{t_2} + y_{u_1}y_{t_3} + y_u y_{b_1} \\
>& 0.
\end{aligned}$$

Since $G[X]$ is $K_{1,3}$-free and $3K_2$-free, we know $G[B^+] \in \{K_{1,2}, K_2, K_3, \varnothing\}$. Combined with $N_B(u_1) \leqslant 1$ we note that $G''[X^+]$ is a proper subgraph of two disjoint K_3 or a P_7 or a disjoint union of a K_3 and a P_4. And $G''[Y^+]$ is empty. Hence G'' is a proper subgraph of $K^*_{a,b}$ or $K^{t_2}_{a,b}$ or $K^{t_3}_{a,b}$. Combined with (1) we have $\lambda(G) \leqslant \lambda(G') < \lambda(G'') < \lambda(K^*_{\lfloor n/2\rfloor,\lceil n/2\rceil})$, contradiction.

Case 3. $G[A^+] \in \bigcup_{i=9}^{11} \mathcal{A}_i$. By Claim 3.1, we know $e(A) \geqslant 7$, then there exists a star $K_{1,t}$ in $G[A^+]$ with $t \geqslant 3$. Similiar with case 1, we can prove $|N_B(u_1)| \leqslant 1$.

Note that there exists a vertex u_2 in $N_A(u_1)$ with $d_A(u_2) \geqslant 3$, suppose $N_A(u_2) = \{u_1, a_1, a_2, \ldots\}$. Since uu_1u_2 form a triangle at u_2, when $|N_B(u_2)| \geqslant 2$, avoid F_3 at u_2, we have

$$e(\{a_1, a_2\}, N_B(u_2)) \leqslant |N_B(u_2)|.$$

Then we have $e(A,B)+2e(A) \leqslant |A|\,|B|-2\,|B|+2\,|A|+|N_B(u_1)|+4$, which contradicts with (4).

When $|N_B(u_2)| \leqslant 1$, we have $e(A,B)+2e(A) \leqslant |A|\,|B|+2\,|A|-2\,|B|+5+|N_B(u_1)|$, which contradics with (4).

Case 4. $G[A^+] \in \mathcal{A}_6$. Since $G[A^+]$ is a C_5 with m chords, from Claim 3.1 we have $m \geqslant 2$. Then there must exist a vertex v_k in A with $d_A(v_k) \geqslant 3$, let v_1, v_2, v_3 be three neighbors of v_k in A.

Since uv_kv_1 form a triangle, when $|N_B(v_k)| \geqslant 3$ avoid F_3 at v_k, we have $e(\{v_2,v_3\}, N_B(v_k)) \leqslant |N_B(v_k)|$, then $e(A,B)+2e(A) \leqslant |A|\,|B|-|B|+20$, when $n>30$, we have $|A|\,|B|-|B|+20 < |A|\,|B|-2\,|B|+2\,|A|+6$, contradicts with (4).

When $|N_B(v_k)| \leqslant 2$, we have $e(A,B)+2e(A) \leqslant |A|\,|B|-|B|+22$, when $n>30$, which contradics with (4).

Case 5. $G[A^+] \in \bigcup_{i\in\{2,4,5\}} \mathcal{A}_i$. Note that $e(B) \leqslant 6$, which contradics with Claim 3.1. □

From Claim 3.3 we know $G[A^+] \in \bigcup_{i\in\{7,8\}} \mathcal{A}_i$. Denote the two centers of star by u_1, u_2, the independent vertex in B by $\{b_i : 1 \leqslant i \leqslant |B|\}$. Let $X = \{u,u_1,u_2\} \cup B$, $X^+ = \{v \in X \mid d_X(v) \geqslant 1\}$, $Y = A-u_1-u_2$, $Y^+ = \{v \in Y \mid d_Y(v) \geqslant 1\}$, $N_A(u_1)-u_2 = S$, $N_A(u_2)-u_1 = T$, $|S| = s$, $|T| = t$. Denote the vertex of S and T by v_{s_i} and v_{t_i} respectively. Easily note that $s,t \leqslant |A|-2$.

Claim 3.4. *If $n>360$ and $G[A^+] \in \bigcup_{i\in\{7,8\}} \mathcal{A}_i$, then $|N_B(u_1)| \leqslant 2$, $|N_B(u_2)| \leqslant 2$.*

Proof. If $G[A^+] \in \bigcup_{i\in\{7,8\}} \mathcal{A}_i$. Suppose on the contrary $|N_B(u_2)| \geqslant 3$.

Case 1. $s,t \geqslant 3$. Avoid F_3 center at u_1, we have $e(N_B(u_1),S) \leqslant \max\{s, |N_B(u_2)|\}$, then $e(A,B)+2e(A) \leqslant |A|\,|B|-2\,|B|+|N_B(u_2)|+2\,|A|+1$. Which contradicts with (4) when $|N_B(u_2)| \leqslant 3$.

When $|N_B(u_2)| \geqslant 4$, avoid F_3 at u_2, we have $e(N_B(u_2),T) \leqslant \max\{|N_B(u_2)|, t\}$, then $e(A,B)+2e(A) \leqslant |A|\,|B|-2\,|B|+2\,|A|+5$, which contradiction with (4).

Case 2. $s \leqslant 2$ and $t \leqslant 2$, $e(A) \leqslant 6$, contradicts with Claim 3.1.

Case 3. $s \leqslant 2$ and $t \geqslant 3$. Firstly we claim $|N_B(u_2)| \leqslant 1$, otherwise suppose $|N_B(u_2)| \geqslant 2$, avoid F_3 at u_2, we have $e(N_B(u_2),T) \leqslant \max\{t, |N_B(u_2)|\}$, then $e(A,B)+2e(A) \leqslant |A|\,|B|-|B|+|A|+7$, contradicts with (4), hence $|N_B(u_2)| \leqslant 1$.

Subcase 3.1. $\alpha'(G[B^+]) = 0$. By the maximality of $\lambda(G)$, the edges between B and Y are complete. For $v \in B$, when $n > 360$ by Lemma 2.6 we have $x_v \geqslant 1-\frac{2}{\lambda} > 1-\frac{4}{n} > \frac{89}{90}$. When $s=2$, let $G' = G-u_1v_{s_1}-u_1v_{s_2}+b_1b_2+b_2b_3+b_3b_4+ub_1$, we can get $\lambda(G')-\lambda(G) > 0$. In G', note that $G'[X^+-u_1]$ is a subgraph of P_7, $G'[Y^++u_1]$ is empty, hence G' is a proper subgraph of $K_{a,b}^{t_2}$. Combined with (1), we have $\lambda(G) < \lambda(G') \leqslant \lambda(K^*_{\lfloor n/2\rfloor,\lceil n/2\rceil})$, contradiction.

When $s=1$, let $G' = G-u_1v_{s_1}+b_1b_2+b_2b_3+b_3b_4+ub_1$, we can get $\lambda(G')-\lambda(G) > 0$. In G', note that $G'[X^+-u_1]$ is a subgraph of P_7 and $G'[Y^++u_1]$ is empty, hence G' is a proper subgraph of $K_{a,b}^{t_2}$. Combined (1), we have $\lambda(G) < \lambda(G') < \lambda(K^*_{\lfloor n/2\rfloor,\lceil n/2\rceil})$, contradiction.

Subcase 3.2. $\alpha'(G[B^+]) \geqslant 2$. Let there exists two disjoint edges c_1c_2, c_3c_4 in $G[B^+]$. Since $v_{t_i}u_2u$ form a triangle, avoid F_3 at v_{t_i}, v_{t_i} can't be adjacent to all of $\{c_1,c_2,c_3,c_4\}$, then we have $e(T,\{c_1,c_2,c_3,c_4\}) \leqslant 3t$, which leads to $e(A,B)+2e(A) \leqslant |A|\,|B|-|B|+|N_B(u_2)|+|A|+4$, which contradics with (4).

Subcase 3.3. $\alpha'(G[B^+]) = 1$. We claim $G[X - u_1]$ is $K_{1,3}$-free, suppose on the contrary there exists a $K_{1,3}$ in $G[X - u_1]$, denote the center vertex of the $K_{1,3}$ by d_1, three vertex in $N_{X-u_1}(d_1)$ by d_2, d_3, d_4. Since $|N_B(u_2)| \leqslant 1$ and $|N_X(d_1)| \geqslant 3$, $d_1 \in X - u_1 - u_2$.

When $|N_Y(d_1)| \geqslant 3$. Avoid F_3 at d_1, there are no $3K_2$ between $N_Y(d_1)$ and $\{d_2, d_3, d_4\}$. Then there exists $i \in \{2, 3, 4\}$, satisfy $e(d_i, N_Y(d_1)) \leqslant 2$ and $e(d_1, Y) + e(d_i, Y) \leqslant |Y| + 2$. When $|N_Y(d_1)| \leqslant 2$. For $i \in \{2, 3, 4\}$, we have $e(d_i, N_Y(d_1)) \leqslant 2$ and $e(d_1, Y) + e(d_i, Y) \leqslant |Y| + 2$.

Let d_2 be a vertex in $N_X(d_1)$, satisfy $e(d_2, N_Y(d_1)) \leqslant 2$ and $e(d_1, Y) + e(d_2, Y) \leqslant |Y| + 2$. Then we have $e(A, B) + 2e(A) = e(u_1, B) + e(u_2, B) + e(Y, B) + 2e(A)$.

If $d_2 \in B$, we have $e(A, B) + 2e(A) \leqslant |A|\,|B| - |B| + |A| + 7$, which leads to a contradiction with (4). If $d_2 = u_2$, we have $e(A, B) + 2e(A) \leqslant |A|\,|B| - |B| + |A| + 7$, which leads to a contradiction with (4). Hence $G[X - u_1]$ is $K_{1,3}$-free.

When $s = 2$, we claim $e(\{u_1, v_{s_1}, v_{s_2}\}, B) \geqslant 2\,|B| + 2$. Suppose on the contrary

$$e(\{u_1, v_{s_1}, v_{s_2}\}, B) \leqslant 2\,|B| + 1,$$

then we have $e(A, B) + 2e(A) \leqslant |A|\,|B| - 2\,|B| + 7 + |N_B(u_2)| + 2(|A| - 2)$, which contradics with (4). Hence $e(\{u_1, v_{s_1}, v_{s_2}\}, B) \geqslant 2\,|B| + 2$. Then there are two triangles $u_1 p_1 v_{s_1}$ and $u_1 p_2 v_{s_2}$ at u_1 where $p_1, p_2 \in B$.

When $s = 1$, we claim $e(\{u_1, v_{s_1}\}, B) \geqslant |B| + 2$. Suppose on the contrary $e(\{u_1, v_{s_1}\}, B) \leqslant |B| + 1$, then we have $e(A, B) + 2e(A) \leqslant |A|\,|B| - 2\,|B| + 7 + |N_B(u_2)| + 2(|A| - 2)$, which contradics with (4). Hence $e(\{u_1, v_{s_1}\}, B) \geqslant |B| + 2$. Then there is a triangle $u_1 p_1 v_{s_1}$ at u_1 where $p_1 \in B$.

Subcase 3.3.1. $G[B^+] = P_i$ ($i = 2$ or 3). Denote the vertex of the P_3 by c_1, c_2, c_3 in order. By the maximality of $\lambda(G)$, the edges between $X - u_1$ and Y are complete. For $v \in B$, when $n > 360$ by Lemma 2.6 we have $x_v \geqslant 1 - \frac{2}{\lambda} > 1 - \frac{4}{n} > \frac{89}{90}$.

When $s = 2$, note that $u_1 \nsim u_2$, otherwise there are three triangles uu_1u_2, $u_1 p_1 v_{s_1}$ and $u_1 p_2 v_{s_2}$ at u_1 where $p_1, p_2 \in B$. Since $G[X - u_1]$ is $K_{1,3}$-free, the vertex in $N_B(u_2)$ not adjacent to all of $\{c_1, c_3\}$, without loss of generality we suppose $N_B(u_2) \nsim c_1$. Let $G' = G - u_1 v_{s_1} - u_1 v_{s_2} + ub_1 + b_1 c_1 + u_1 u_2$, we have $\lambda(G') - \lambda(G) > 0$. Since $|N_B(u_2)| \leqslant 1$, $G'[X^+ - u_1]$ is a subgraph of a P_7 or a C_6, and $G'[Y^+ + u_1]$ is empty. Hence G' is a subgraph of $K_{a,b}^{t_2}$ or $K_{a,b}^{6}$. Combined with (1) we have $\lambda(G) < \lambda(G') \leqslant \lambda(K^*_{\lfloor n/2 \rfloor, \lceil n/2 \rceil})$, contradiction.

When $s = 1$. Since $G[X - u_1]$ is $K_{1,3}$-free, the vertex in $N_B(u_2)$ not adjacent to all of $\{c_1, c_3\}$, without loss of generality we suppose $N_B(u_2) \nsim c_1$. Let $G' = G - u_1 v_{s_1} + ub_1 + b_1 c_3$, we have $\lambda(G') - \lambda(G) > 0$. Since $|N_B(u_2)| \leqslant 1$, $G'[X^+ - u_1]$ is a subgraph of a P_7 or a C_6, and $G'[Y^+ + u_1]$ is empty. Hence G' is a subgraph of $K_{a,b}^{t_2}$ or $K_{a,b}^{6}$. Combined with (1) we have $\lambda(G) < \lambda(G') \leqslant \lambda(K^*_{\lfloor n/2 \rfloor, \lceil n/2 \rceil})$, contradiction.

Subcase 3.3.2. $G[B^+] = K_3$. Denote the vertex of $G[B^+]$ by t_1, t_2, t_3. By the maximality of $\lambda(G)$, the edges between $X - u_1$ and Y are complete. For $v \in B$, when $n > 360$ by Lemma 2.6 we have $x_v \geqslant 1 - \frac{2}{\lambda} > 1 - \frac{4}{n} > \frac{89}{90}$.

When $s = 2$, there are two triangles $u_1 p_1 v_{s_1}$ and $u_1 u v_{s_2}$ at u_1 where $p_1 \in B$. Then u_1 not adjacent to two of $\{t_1, t_2, t_3\}$ and $u_1 \nsim u_2$, otherwise there will be an F_3 at u_1. Without loss of generality, we assume u_1 not adjacent to $\{t_1, t_2\}$. Let $G' = G - u_1 v_{s_1} - u_1 v_{s_2} +$

$u_1t_1 + u_1t_2 + u_1u_2$, we have

$$\begin{aligned}\lambda(G') - \lambda(G) &= x^T(A(G') - A(G))x \\ &= -(x_{u_1}x_{v_{s_1}} + x_{u_1}x_{v_{s_2}}) + x_{u_1}x_{t_1} + x_{u_1}x_{t_2} + x_{u_1}x_{u_2} \\ &> -2 + 3 \cdot (\frac{89}{90})^2 \\ &> 0.\end{aligned}$$

Since $|N_B(u_2)| \leqslant 1$, $G'[X^+ - u_1]$ is a subgraph of two disjoint K_3, and $G'[Y^+ + u_1]$ is empty. Hence G' is a subgraph of $K^*_{a,b}$. Then we have $\lambda(G) < \lambda(G') < \lambda(K^*_{\lfloor n/2\rfloor,\lceil n/2\rceil})$, contradiction.

When $s = 1$. Avoid F_3 at u_1, we claim u_1 not adjacent to all of $\{u_2, t_1, t_2, t_3\}$. Suppose on the contrary u_1 adjacent to all of $\{u_2, t_1, t_2, t_3\}$, then $u_1p_1v_{s_1}$, u_1u_2u and $u_1t_1t_2$ form an F_3 at u_1 where $p_1 \in B$, contradiction. Without loss of generality, we assume $u_1 \nsim u_2$. Let $G' = G - u_1v_{s_1} + u_1u_2 + ub_1$, we have $\lambda(G') - \lambda(G) > 0$.

Since $|N_B(u_2)| \leqslant 1$, $G'[X^+ - u_1]$ is a subgraph of a disjoint union of K_3 and P_4, and $G'[Y^+ + u_1]$ is empty. Hence G' is a subgraph of $K^{t_1}_{a,b}$. Then we have $\lambda(G) < \lambda(G') \leqslant \lambda(K^*_{\lfloor n/2\rfloor,\lceil n/2\rceil})$, contradiction. □

Claim 3.5. *If $G[A^+] \in \bigcup_{i\in\{7,8\}} \mathcal{A}_i$, then $\alpha'(G[X]) \leqslant 3$. Furthermore, we have $\alpha'(G[B]) \leqslant 2$.*

Proof. Assume on the contrary $\alpha'(G[X]) \geqslant 4$, denoted the vertex of 4 disjoint edges by $\{c_1c_2, c_3c_4, c_5c_6, c_7c_8\}$, the vertex of $S \cap T$ by $\{v_{r_i}\}$. Since $uu_1v_{r_i}$ form a triangle, avoid F_3 at r_i, we have $e(S\cap T, \{c_i\}_{i=1}^8) \leqslant 6\,|S \cap T|$, then $e(A, B) + 2e(A) \leqslant |A|\,|B| - 2\,|B| + 6 + 2\,|A|$, which leads a contradiction with (4). Hence $\alpha'(G[X]) \leqslant 3$.

Furthermore, we can prove $\alpha'(G[B]) \leqslant 2$. Suppose on the contrary $\alpha'(G[B]) \geqslant 3$. Note that uu_1 is an edge in $X - B$, which leads to $\alpha'(G[X]) \geqslant 4$, contradiction. □

Claim 3.6. *If $n > 360$ and $G[A^+] \in \bigcup_{i\in\{7,8\}} \mathcal{A}_i$, then $G[X]$ is $K_{1,7}$-free.*

Proof. Suppose on the contrary $G[X]$ contain a $K_{1,p}$ ($p \geqslant 7$), denote the center of the $K_{1,p}$ by c_1, the vertex of $N_Y(c_1)$ by $\{a_i : 0 \leqslant i \leqslant |N_Y(c_1)|\}$, the vertex of $N_X(c_1)$ by $\{c_1, \ldots, c_{p+1}\}$. Since $|N_B(u_1)| \leqslant 2$, $|N_B(u_2)| \leqslant 2$ and $|N_X(c_1)| \geqslant 7$, $c_1 \in B = X - u - u_1 - u_2$.

When $|N_Y(c_1)| \geqslant 3$. Avoid F_3 at c_1, there not exist $3K_2$ between $N_X(c_1)$ and $N_Y(c_1)$. For $i \in \{4, 5, 6, 7, 8\}$, we have $e(c_i, N_Y(c_1)) \leqslant 2$ and $e(c_1, Y) + e(c_i, Y) \leqslant |Y| + 2$. When $|N_Y(c_1)| \leqslant 2$. For $i \in \{4, 5, 6, 7, 8\}$, we have $e(c_i, N_Y(c_1)) \leqslant 2$ and $e(c_1, Y) + e(c_i, Y) \leqslant |Y| + 2$.

Firstly we claim that there not exist a $K_{1,3}$ in $G[X - c_1]$, suppose on the contrary suppose there exists a $K_{1,3}$ in $G[X - c_1]$ with center denoted by d_1, the vertex of $N_X(d_1)$ denoted by d_2, d_3, d_4.

When $|N_Y(d_1)| \geqslant 3$. Avoid F_3 at d_1, there not exist $3K_2$ between $N_X(d_1)$ and $N_Y(d_1)$. Then there must exist $i \in \{2, 3, 4\}$, satisfy $e(d_i, N_Y(d_1)) \leqslant 2$ and $e(d_1, Y) + e(d_i, Y) \leqslant |Y| + 2$. When $|N_Y(d_1)| \leqslant 2$ for $i \in \{2, 3, 4\}$, we also have $e(d_i, N_Y(d_1)) \leqslant 2$ and $e(d_1, Y) + e(d_i, Y) \leqslant |Y| + 2$.

Let c_4 be a vertex in $N_X(c_1)$ and d_2 be a vertex in $N_X(d_1)$, satisfy $c_1, c_4 \in X - c_1 - c_4 - d_1 - d_2$, $e(c_1, Y) + e(c_4, Y) \leqslant |Y| + 2$ and $e(d_1, Y) + e(d_2, Y) \leqslant |Y| + 2$. Then we have $e(A, B) + 2e(A) \leqslant e(u_1, B) +$

$e(u_2,B)+e(Y,c_1)+e(N_Y(c_1),c_4)+e(Y-N_Y(c_1),c_4)+e(Y,B-c_1-c_4)+2e(Y,u_1)+2e(Y,u_2)+2$.

If $d_1,d_2\in B$, we have $e(A,B)+2e(A)\leqslant |A|\,|B|+2\,|A|-2\,|B|+6$, contradiction with (4). If $d_1=u_1,d_2\in B$, we have $e(A,B)+2e(A)\leqslant |A|\,|B|+2\,|A|-2\,|B|+6$, contradiction with (4). If $d_2=u_1,d_1\in B$, we have $e(A,B)+2e(A)\leqslant |A|\,|B|+2\,|A|-2\,|B|+6$, contradiction with (4). If $d_1=u_1,d_2=u_2$, since $e(N_Y(d_1),d_2)\leqslant 2$ we have $|N_Y(d_1)|+e(Y,d_2)\leqslant |Y|+2$, which leads to $e(A,B)+2e(A)\leqslant |A|\,|B|+|A|-2\,|B|+10$, contradiction with (4). Hence $G[X-c_1]$ is $K_{1,3}$-free.

From Claim 3.5 we have $\alpha'(G[X])\leqslant 3$. Since $G[X]$ contain a $K_{1,p}$ ($p\geqslant 7$) and $G[X-c_1]$ is $K_{1,3}$-free, we deduce $G[B^+-c_1]\in\{K_{1,2},K_2,K_3,\varnothing\}$.

Case 1. $|N_Y(c_1)|\geqslant 3$.

Avoid F_3 at c_1, there not exist $3K_2$ between $N_X(c_1)$ and $N_Y(c_1)$. Then there must exist subset $\{c_4,\ldots,c_{p+1}\}$ and $\left\{a_3,\ldots,a_{|N_Y(c_1)|}\right\}$ of $N_X(c_1)$ and $N_Y(c_1)$, satisfy there are no edges between $\{c_4,\ldots,c_{p+1}\}$ and $\left\{a_3,\ldots,a_{|N_Y(c_1)|}\right\}$. We claim

$$|N_Y(c_1)|<\frac{|A|}{4}, \tag{5}$$

otherwise suppose on the contrary $|N_Y(c_1)|\geqslant\frac{|A|}{4}$. Avoid F_3 at c_1, we have $e(N_Y(c_1),N_X(c_1))\leqslant\max\{2\,|N_Y(c_1)|\,,2p\}$. Then we have

$$e(A,B)+2e(A)\leqslant\max\{2\,|N_Y(c_1)|\,,2p\}+2+2s+2t+2. \tag{6}$$

When $|N_Y(c_1)|\geqslant p$, we have $-p\,|N_Y(c_1)|+\max\{2\,|N_Y(c_1),2p|\}\leqslant -|A|$. When $|N_Y(c_1)|<p$, we have $-p\,|N_Y(c_1)|+\{2\,|N_Y(c_1)|\,,2p\}\leqslant -|A|$. Back to (6), we have $e(A,B)+2e(A)\leqslant |A|\,|B|-2\,|B|-2\,|A|+|N_B(u_1)|+|N_B(u_2)|+4+4(|A|-2)$, which contradics with (4), hence $|N_Y(c_1)|<\frac{|A|}{4}$.

Then we have $\lambda x_{c_1}=\sum_{i=2}^{p+1}x_{c_i}+\sum_{i\in N_Y(c_1)}x_i\leqslant |N_Y(c_1)|+p<\frac{|A|}{4}+p$.

By the maximality of $\lambda(G)$ and $\alpha'(G[X])\leqslant 3$, we know $B-c_1$ adjacent to all vertex in $A-u_1-u_2-N_Y(c_1)$. By Claim 3.1 when $n>360$ we know $|A|<\frac{7}{12}n$ and $\lambda>\frac{n}{2}$. For $i\in B-c_1$, we can get

$$\lambda x_i>\frac{17}{28}\,|A|-2. \tag{7}$$

Then we have

$$\begin{aligned}\sum_{i=4}^{p+1}\lambda x_{c_i}-\lambda x_{c_1}&\geqslant (p-2)(\frac{17}{28}\,|A|-2)-p-\frac{|A|}{4}\\&\geqslant 5(\frac{17}{28}\,|A|-2)-7-\frac{|A|}{4}>0.\end{aligned} \tag{8}$$

From (8) when $j\geqslant 3$, we have

$$-x_{c_1}x_{a_j}+\sum_{i=4}^{p+1}x_{c_i}x_{a_j}>0. \tag{9}$$

Subcase 1.1. $G[B^+-c_1]\neq\varnothing$. Avoid F_3 at a_1 (or a_2), a_1 (or a_2) not adjacent at least one vertex of $\{u_1,u_2\}$ or $N_B(c_1)$ or $G[B^+-c_1]$. Without loss of generality, suppose $x_{a_1}\geqslant x_{a_2}$, assume $a_1\nsim v_1, a_2\nsim v_2$

where $v_1, v_2 \in B - c_1$. From (7), we know $\lambda x_{v_1} \geqslant \lambda - 2 - \frac{|A|}{4}$ and $\lambda x_{v_2} \geqslant \lambda - 2 - \frac{|A|}{4}$. Then we have

$$-x_{c_1}x_{a_1} - x_{c_1}x_{a_2} + x_{c_1}x_u + x_{a_1}x_{v_1} + x_{a_2}x_{v_2} > 0. \tag{10}$$

Let $G' = G + \{-c_1a_j + \{c_ia_j : 4 \leqslant i \leqslant p+1\} : 3 \leqslant j \leqslant |N_Y(c_1)|\} - c_1a_1 - c_1a_2 + c_1u + a_1v_1 + a_2v_2$. From (9) and (10), we get $\lambda(G') > \lambda(G)$.

From Claim 3.5 we have $\alpha'(G[X]) \leqslant 3$. Since $G[X - c_1]$ is $K_{1,3}$-free and $G[B^+ - c_1] \in \{K_{1,2}, K_2, K_3, \varnothing\}$. In G', $G'[X^+ - c_1]$ is a subgraph of two disjoint K_3 or a C_5, and $G'[Y^+ + c_1]$ is empty. Hence G' is a subgraph of $K^*_{a,b}$ or $K^5_{a,b}$. Then we have $\lambda(G) < \lambda(G') \leqslant \lambda(K^*_{\lfloor n/2\rfloor,\lceil n/2\rceil})$, contradiction.

Subcase 1.2. $G[B^+ - c_1] = \varnothing$.

Let $G' = G + \{-c_1a_j + \{c_ia_j : 4 \leqslant i \leqslant p+1\} : 3 \leqslant j \leqslant |N_Y(c_1)|\}$, G'' is obtained from G' by adding edges between $X - c_1$ and $Y + c_1$ except uc_1. Let $x'' = (x''_1, x''_2, \ldots, x''_n)^T$ be a Perron vector of G'', normalize x'' so that $x''_u = \max\{x''_v : v \in G''\} = 1$. Then from (9) we have $\lambda(G') > \lambda(G)$. Since G' is a subgraph of G'', we have $\lambda(G'') \geqslant \lambda(G')$.

When the corresponding to maximum component of x'' in $X - c_1$, by Lemma 2.6 we have $x''_i \geqslant 1 - \frac{2}{\lambda(G'')} > 1 - \frac{4}{n} > \frac{89}{90}$ for $i \in X - c_1$.

When the corresponding to maximum component of x'' in $Y + c_1$, by Lemma 2.6 we have $x''_i \geqslant 1 - \frac{2}{\lambda(G'')} > 1 - \frac{4}{n} > \frac{89}{90}$ for $i \in Y + c_1$. Note that $\lambda(G'') < \frac{7}{12}n$. For $i \in X - c_1$, we have $x''_i > \frac{4}{5}$.

If $|N_B(u_1)| \neq 0$, let $u_3 \in N_B(u_1)$. Let $G''' = G'' - c_1a_1 - c_1a_2 - u_1u_3 + uc_1 + b_1b_2 + b_2b_3 + b_3b_1$, we can get $\lambda(G''') > \lambda(G'')$.

From Claim 3.5 we have $\alpha'(G[X]) \leqslant 3$. Since $G[X - c_1]$ is $K_{1,3}$-free and $G[B^+ - c_1] \in \{K_{1,2}, K_2, K_3, \varnothing\}$. In G''', $G'''[X^+ - c_1]$ is a subgraph of two disjoint K_3 or a disjoint union of K_3 and P_4, $G'''[Y^+ + c_1]$ is empty. Hence G''' is a subgraph of $K^*_{a,b}$ or $K^{t_1}_{a,b}$, then $\lambda(G) < \lambda(G''') \leqslant \lambda(K^*_{a,b})$, contradiction.

Case 2. $|N_Y(c_1)| \leqslant 2$.

Subcase 2.1. $G[B^+ - c_1] \neq \varnothing$. Avoid F_3 at a_1 (or a_2), a_1 (or a_2) not adjacent at least one vertex of $\{u_1, u_2\}$ or $N_B(c_1)$ or $G[B - c_1]$. Without loss of generality, suppose $x_{a_1} \geqslant x_{a_2}$, assume $a_1 \nsim v_1$, $a_2 \nsim v_2$,

Let G' be obtained from G by adding edges between $X - c_1$ and $Y + c_1$ except uc_1, a_1v_1, a_1v_2, and $x' = (x'_1, x'_2, \ldots, x'_n)^T$ is a Perron vector of G', normalize x' so that $x'_u = \max\{x'_v : v \in G'\} = 1$. Note that G is a subgraph of G', then $\lambda(G') \geqslant \lambda(G)$.

When the corresponding to maximum component of x' in $X - c_1$, by Lemma 2.6, we have $x'_i \geqslant 1 - \frac{2}{\lambda(G')} > 1 - \frac{4}{n} > \frac{89}{90}$ for $i \in X - c_1$.

When the corresponding to maximum component of x' in $Y + c_1$, by Lemma 2.6, we have $x'_i \geqslant 1 - \frac{2}{\lambda(G')} > 1 - \frac{4}{n} > \frac{89}{90}$ for $i \in Y + c_1$. Note that $\lambda(G') < \frac{7}{12}n$. For $i \in X - c_1$, we have $x'_i > \frac{4}{5}$.

Let $G'' = G' - c_1a_1 - c_1a_2 + a_1v_1 + a_1v_2 + uc_1$, from above we can get $\lambda(G'') > \lambda(G')$.

From Claim 3.5 we have $\alpha'(G[X]) \leqslant 3$. Since $G[X - c_1]$ is $K_{1,3}$-free and $G[B^+ - c_1] \in \{K_{1,2}, K_2, K_3, \varnothing\}$. In G'', $G''[X^+ - c_1]$ is a subgraph of two disjoint K_3 or a C_5, and $G'[Y^+ + c_1]$ is empty. Hence G' is a subgraph of $K^*_{a,b}$ or $K^5_{a,b}$, then $\lambda(G) < \lambda(G'') \leqslant \lambda(K^*_{a,b})$, contradiction.

Subcase 2.2. $G[B^+ - c_1] = \varnothing$.

Let G' be obtained from G by adding edges between $X - c_1$ and $Y + c_1$ except uc_1, and $x' = (x'_1, x'_2, \ldots, x'_n)^T$ is a Perron vector of G', normalize x' so that $x'_u = \max\{x'_v : v \in G'\} = 1$. Note that G is a subgraph of G', then $\lambda(G') \geqslant \lambda(G)$.

When the corresponding to maximum component of x' in $X - c_1$, by Lemma 2.6, we have $x'_i \geqslant 1 - \frac{2}{\lambda(G')} > 1 - \frac{4}{n} > \frac{89}{90}$ for $i \in X - c_1$.

When the corresponding to maximum component of x' in $Y + c_1$, by Lemma 2.6, we have $x'_i \geqslant 1 - \frac{2}{\lambda(G')} > 1 - \frac{4}{n} > \frac{89}{90}$ for $i \in Y + c_1$. Note that $\lambda(G') \leqslant \frac{7}{12}n$. For $i \in X - c_1$, we have $x'_i > \frac{4}{5}$.

If $|N_B(u_1)| \neq 0$, let $u_3 \in N_B(u_1)$ and $G'' = G' - c_1a_1 - c_1a_2 - u_1u_3 + uc_1 + b_1b_2 + b_2b_3 + b_3b_1$, from above we can get $\lambda(G'') > \lambda(G)$.

From Claim 3.5 we have $\alpha'(G[X]) \leqslant 3$. Since $G[X - c_1]$ is $K_{1,3}$-free and $G[B^+ - c_1] \in \{K_{1,2}, K_2, K_3, \varnothing\}$. In G'', $G''[X^+ - c_1]$ is a subgraph of two disjoint K_3 or a disjoint union of K_3 and P_4, $G''[Y^+ + c_1]$ is empty. Hence G'' is a subgraph of $K^*_{a,b}$ or $K^{t_1}_{a,b}$, then $\lambda(G) < \lambda(G'') \leqslant \lambda(K^*_{a,b})$, contradiction. □

From Claim 3.5 and 3.6 we have $\alpha'(G[B]) \leqslant 2$ and $\Delta(G[B]) \leqslant 6$, combined with Lemma 2.4 we can give an upper bound of $e(B)$ which will be useful in the proof of Claim 3.7.

Claim 3.7. *If $n > 360$ and $G[A^+] \in \bigcup_{i\in\{7,8\}} \mathcal{A}_i$, then $G[X]$ not contain a $K_{1,3}$ or a $3K_2$ as a subgraph.*

Proof. Suppose on the contrary $G[X]$ contain a $K_{1,3}$, denote the center of the $K_{1,3}$ by c_1, the leaf vertex by c_2, c_3, c_4 (When $G[X]$ contrain a $3K_2$, the proof is similiar). Avoid F_3 at c_1, there exists $i \in \{2, 3, 4\}$, satisfy $e(c_1, Y) + e(c_2, Y) \leqslant |Y| + 2$. By Kelmans transformation we know when $e(c_2, Y) \leqslant 2$, G get maximum spectral radius.

Let the graph G' be obtained from G by adding $\frac{1}{2}|Y|$ edges between c_2 and Y, adding edges between $X - c_2$ and Y. Let the graph G'' be obtained from G' by adding all edges between c_2 and Y, deleting edges in B and edges between B and $\{u_1, u_2\}$. Since G is a subgraph of G' and G'' is a subgraph of $K^*_{a,b}$, we have $\lambda(G) < \lambda(G')$ and $\lambda(G'') < \lambda(K^*_{\lfloor n/2\rfloor,\lceil n/2\rceil})$. Then it suffices to prove $\lambda(G'') > \lambda(G')$. Let $y = (y_1, \ldots, y_n)^T$ be a Perron vector of graph G', normalize y so that $y_u = \max\{y_v : v \in G'\} = 1.$.

Case 1. The vertex corresponding to maximum component of y is in X. From Claim 3.6 we know $G'[X]$ is $K_{1,7}$-free. For $i \in X - c_2$, by Lemma 2.6 we know $y_i \geqslant 1 - \frac{6}{\lambda(G')} > 1 - \frac{12}{n}$. From Claim 3.1 when $n > 360$, we have $|B| > \frac{5}{12}n$. Note that $\lambda(G') \leqslant \Delta(G') \leqslant |Y| + 6 = |A| + 4 < \frac{7}{12}n + 4 < \frac{7}{12}n + \frac{1}{90}n$. By the maximality of $\lambda(G)$, every vertex in Y adjacent all the vertex in $X - c_2$. For $i \in Y$, we can get $y_i > \frac{17}{25}$. From Lemma 2.6 we have $y_{c_2} > \frac{2}{5}$. From Claim 3.5 and 3.6 we have $\alpha'(B) \leqslant 2$ and $G'[B]$ is $K_{1,7}$-free, then from Lemma 2.4 we know $e(B) \leqslant 14$. By Claim 3.4 we have $e(\{u_1, u_2\}, B) \leqslant 4$. When $n > 360$

$$\begin{aligned}\lambda(G'') - \lambda(G') &> y^T(A(G'') - A(G'))y \\ &> \sum_{i\in N_Y(c_2)} y_{c_2}y_i - e(B) - e(\{u_1, u_2\}, B) \\ &> \frac{2}{5}\cdot\frac{17}{25}\cdot\frac{1}{2}(|Y| - 2) - 18 \\ &> 0.\end{aligned}$$

Case 2. The vertex corresponding to maximum component of y is in Y. By the maximality of $\lambda(G)$, every vertex in Y adjacent all the vertex in $X - c_2$. From Claim 3.1 when $n > 360$, we have $|B| > \frac{5}{12}n$.

DOI: 10.4236/am.2026.175017

Note that $\lambda(G') \leqslant \Delta(G') \leqslant |Y| + 6 = |A| + 4 < \frac{7}{12}n + 4 < \frac{7}{12}n + \frac{1}{90}n$. For $i \in Y$, by Lemma 2.6 we have $y_i > 1 - \frac{2}{n} > \frac{179}{180}$ and $y_{c_2} > \frac{2}{5}$. From Claim 3.5 and 3.6 we have $\alpha'(B) \leqslant 2$ and $G[B]$ is $K_{1,7}$-free, then from Lemma 2.4 we know $e(B) \leqslant 14$. By Claim 3.4 we have $e(\{u_1, u_2\}, B) \leqslant 4$. When $n > 360$

$$\begin{aligned}\lambda(G'') - \lambda(G') &> y^T(A(G'') - A(G'))y \\ &> \sum_{i \in N_Y(c_2)} y_{c_2} y_i - e(B) - e(\{u_1, u_2\}, B) \\ &> \frac{2}{5} \cdot \frac{179}{180} \cdot \frac{1}{2}(|Y| - 2) - 18 \\ &> 0.\end{aligned}$$

Hence $\lambda(G) < \lambda(G') < \lambda(G'') < \lambda(K^*_{\lfloor n/2 \rfloor, \lceil n/2 \rceil})$, contradiction. □

From Claim 3.7 we have $G[X]$ is $K_{1,3}$-free and $3K_2$-free, then $\alpha'(G[B^+]) \leqslant 1$ and $G[B^+] \in \{K_{1,2}, K_2, K_3, \varnothing\}$. Easily note that $G[X^+]$ is a subgraph of two disjoint K_3 or a C_p $(3 \leqslant p \leqslant 5)$. Hence G is a subgraph of $K^*_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ or $K^p_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ $(3 \leqslant p \leqslant 5)$, which leads to $\lambda(G) \leqslant \lambda(K^*_{\lfloor n/2 \rfloor, \lceil n/2 \rceil})$ with equality holds if and only if $G = K^*_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. Since $\lambda(G) \geqslant \lambda(K^*_{\lfloor n/2 \rfloor, \lceil n/2 \rceil})$, then we have $G = K^*_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. This complete this proof. □

References

[1] Bondy, J.A. and Murty, U.S.R. (2008) Graph Theory, Graduate Texts in Mathematics, vol. 24. Springer.

[2] Erdos, P., Furedi, Z., Gould, R.J. and Gunderson, D.S. (1995) Extremal Graphs for Intersecting Triangles. *Journal of Combinatorial Theory, Series B*, **64**, 89-100. https://doi.org/10.1006/jctb.1995.1026

[3] Cioab, S., Feng, L., Tait, M. and Zhang, X. (2020) The Maximum Spectral Radius of Graphs without Friendship Subgraphs. *The Electronic Journal of Combinatorics*, **27**, Article No. P4.22. https://doi.org/10.37236/9179

[4] Zhai, M., Liu, R. and Xue, J. (2022) A Unique Characterization of Spectral Extrema for Friendship Graphs. *The Electronic Journal of Combinatorics*, **29**, Article No. P3.32. https://doi.org/10.37236/11183

[5] Li, Y., Lu, L. and Peng, Y. (2023) Spectral Extremal Graphs for the Bowtie. *Discrete Mathematics*, **346**, Article ID: 113680. https://doi.org/10.1016/j.disc.2023.113680

[6] Chen, Y., Li, S., Zhang, L. and Zhang, M. (2025) Spectral Extrema of F_2-Free Graphs with Given Size Revisited. *Discrete Applied Mathematics*, **375**, 294-307. https://doi.org/10.1016/j.dam.2025.06.007

[7] Yu, L., Li, Y. and Peng, Y. (2025) Spectral Extremal Graphs for Fan Graphs. *Discrete Mathematics*, **348**, Article ID: 114391. https://doi.org/10.1016/j.disc.2024.114391

[8] Gao, J. and Li, X. (2026) Spectral Radius of Graphs of Given Size with a Forbidden Fan Graph F_6. *Discrete Mathematics*, **349**, Article ID: 114695. https://doi.org/10.1016/j.disc.2025.114695

[9] Li, S., Zhao, S. and Zou, L. (2025) Spectral Extrema of Graphs with Fixed Size: Forbidden a Fan Graph, a Friendship Graph, or a Theta Graph. *Journal of Graph Theory*, **110**, 483-495. https://doi.org/10.1002/jgt.23287

[10] Zhang, Y. and Wang, L. (2024) On the Spectral Radius of Graphs without a Gem. *Discrete Mathematics*, **347**, Article ID: 114171. https://doi.org/10.1016/j.disc.2024.114171

[11] Li, Y., Feng, L. and Peng, Y. (2025) A Spectral Erdös-Faudree-Rousseau Theorem. *Journal of Graph Theory*, **110**, 408-425. https://doi.org/10.1002/jgt.23280

[12] Hoffman, A.J. and Smith, J.H. (1975) Recent Advances in Graph Theory. Academic Praha.

[13] Csikvri, P. (2009) On a Conjecture of V. Nikiforov. *Discrete Mathematics*, **309**, 4522-4526. https://doi.org/10.1016/j.disc.2009.02.013

[14] Chvtal, V. and Hanson, D. (1976) Degrees and Matchings. *Journal of Combinatorial Theory, Series B*, **20**, 128-138. https://doi.org/10.1016/0095-8956(76)90004-6