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Invariant Quartic Characterization of Delay-Independent Stability in Two-Dimensional Systems with Two Rank-One Delay Channels

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Abstract

Delay-independent stability of systems with several independent delays is generally governed by transcendental characteristic equations and multidimensional phase conditions. This paper derives an exact closed-form specialization for two-dimensional linear retarded systems with two independent discrete delays and delayed coefficient matrices of rank at most one (including zero-rank limiting cases). Using identities specific to 2×2 matrices, the complete characteristic quasipolynomial is represented by seven simultaneous-similarity invariants. Exact elimination of the two unit-modulus delay phases then yields a single explicit monic quartic $F(y)$, $y=\omega^2$, such that the complete realizable crossing-frequency set is $\{\omega>0: F(\omega^2)\le 0\}$. Consequently, stability for every finite delay pair is equivalent to Hurwitz stability of $A+B+C$ and $F(y)>0$ for all $y>0$. Because F' is cubic, the latter condition admits a finite exact decision procedure based on at most three positive stationary points. The quartic also provides a factorization of $F(0)$ through $\det(A\pm B\pm C)$, a global Hermitian positivity characterization, a triangle/Heron geometry when the mixed delayed-channel invariant vanishes, and explicit reconstruction of critical-delay families. The result is positioned as a transparent invariant specialization of established algebraic and frequency-domain delay-independent stability theory rather than as a new general multidelay criterion.

Keywords

Time-Delay Systems, Matrix Invariants, Stability Crossings, Spectral

Analysis, Bifurcation Theory

1. Introduction

Time delays arise naturally in control, communication, biological, mechanical, and networked systems. Even when the undelayed model is finite-dimensional and linear, discrete delays convert the characteristic polynomial into a quasipolynomial with infinitely many roots. A fundamental robustness question is therefore whether asymptotic stability can be guaranteed independently of the delay values.

Necessary and sufficient conditions for delay-independent stability have a long history. Lewis and Anderson studied stability under arbitrary delays for linear autonomous systems with delayed interactions, with a distinction between robust quasi-diagonal-dominance statements and weaker conditions for a fixed system [1]. Cooke and Ferreira developed spectral-limit and global-in-delay stability results for retarded functional differential equations [2], while Hale, Infante and Tsen gave necessary and sufficient all-delay stability conditions for linear retarded and neutral equations [3]. Feng, Liu and Zhao subsequently showed that multidelay delay-independent stability questions can be converted from transcendental to algebraic conditions [4]. Chen and Latchman developed necessary and sufficient frequency-sweeping tests for commensurate and noncommensurate delay structures [5].

For systems with two delays and rank-one delayed matrices, Wu and Ren obtained necessary and sufficient delay-independent stability criteria by reducing the problem to the existence of real roots of auxiliary polynomials [6]. Their work is the closest prior result to the structural class studied here. Gu, Niculescu and Chen characterized stability crossing sets and crossing curves for general scalar quasipolynomials of arbitrary order with two delays [7]. Michiels and Niculescu treated delay-independent stability and delay-interference phenomena for multiple delays, including independently varying and constrained delays [8]. Yu, Zhong and Li also formulated an algebraic criterion for linear differential systems with two delays [9]. Naghnaeian and Gu later studied stability crossing sets for systems with two scalar-delay channels and emphasized the care needed when eliminating mixed exponential terms [10]. Alikoc and Ergenc proposed a general polynomial method based on self-inversive auxiliary polynomials for LTI systems with constant delays [11].

The objective of the present work is not to replace these general theories. Instead, we exploit the exceptional algebra of 2×2 matrices to obtain a compact invariant specialization. For

$$\dot{x}(t) = Ax(t) + Bx(t-\tau_1) + Cx(t-\tau_2),\ \ A,B,C \in \mathbb{R}^{2\times 2},$$

with $\det B = \det C = 0$, the complete characteristic quasipolynomial can be compressed to seven simultaneous-similarity invariants. The two independent phase

variables can then be eliminated analytically, leaving a single quartic polynomial in $y=\omega^2$.

The contribution is deliberately specialized rather than general. Within this two-dimensional class, the paper provides four linked results. First, the complete characteristic quasipolynomial is compressed to seven simultaneous-similarity invariants. Second, both independent delay phases are eliminated exactly, producing a universal monic quartic $F(y)$ whose nonpositive set is the complete realizable crossing-frequency set. Third, delay-independent stability is reduced to zero-delay Hurwitz stability plus strict positivity of this quartic on $(0,\infty)$, with a finite decision test because F' is cubic. Fourth, the same invariant quartic yields structural information that is otherwise obscured: the factorization of $F(0)$ into four signed determinants, a global Hermitian interpretation, a triangle/Heron geometry when the mixed interaction vanishes, explicit critical-delay reconstruction, and algebraic parameter boundaries. Thus, the novelty sought here is the closed invariant form and the resulting transparency of the exact criterion, not the general existence of algebraic or frequency-domain delay-independent stability tests.

2. Problem Formulation and Algebraic Preliminaries

Consider

$$\dot{x}(t)=Ax(t)+Bx(t-\tau_1)+Cx(t-\tau_2),\ x(t)\in\mathbb{R}^2, \tag{1}$$

where $A,B,C\in\mathbb{R}^{2\times 2}$ and $\tau_1,\tau_2\geq 0$ are independent.

The characteristic equation is

$$\Delta(\lambda;\tau_1,\tau_2)=\det\left[\lambda I-A-Be^{-\lambda\tau_1}-Ce^{-\lambda\tau_2}\right]=0. \tag{2}$$

Definition 2.1. *System* (1) *is called delay-independently asymptotically stable if it is asymptotically stable for every finite pair*

$$(\tau_1,\tau_2)\in[0,\infty)^2.$$

For every 2×2 matrix X,

$$\operatorname{adj}(X)=(\operatorname{tr}X)I-X, \tag{3}$$

and

$$\det(X+Y)=\det X+\det Y+\operatorname{tr}(\operatorname{adj}(X)Y). \tag{4}$$

Define

$$a=\operatorname{tr}A,\quad b=\operatorname{tr}B,\quad c=\operatorname{tr}C,$$

$$\alpha=\det A,\quad \beta=\det B,\quad \gamma=\det C,$$

and

$$p=\operatorname{tr}(\operatorname{adj}(A)B),\quad q=\operatorname{tr}(\operatorname{adj}(A)C),\quad r=\operatorname{tr}(\operatorname{adj}(B)C).$$

Equivalently,

$$p=ab-\operatorname{tr}(AB),\quad q=ac-\operatorname{tr}(AC),\quad r=bc-\operatorname{tr}(BC).$$

These quantities are invariant under simultaneous similarity

$$(A,B,C)\mapsto\left(S^{-1}AS,S^{-1}BS,S^{-1}CS\right).$$

Throughout the main results we impose

$$\det B=\det C=0, \tag{5}$$

so each delayed matrix has rank at most one.

3. Invariant Characteristic Representation

Proposition 3.1. *Under* (5), *the characteristic quasipolynomial is*

$$\boxed{\Delta(\lambda)=\lambda^2-a\lambda+\alpha+(p-b\lambda)\mathrm{e}^{-\lambda\tau_1}+(q-c\lambda)\mathrm{e}^{-\lambda\tau_2}+r\mathrm{e}^{-\lambda(\tau_1+\tau_2)}.} \tag{6}$$

Proof. Let $z_1=\mathrm{e}^{-\lambda\tau_1}$ and $z_2=\mathrm{e}^{-\lambda\tau_2}$. For a 2×2 matrix $H=A+Bz_1+Cz_2$,

$$\det(\lambda I-H)=\lambda^2-\lambda\operatorname{tr}H+\det H.$$

Now

$$\operatorname{tr}H=a+bz_1+cz_2,$$

while repeated use of (4) gives

$$\det H=\alpha+pz_1+qz_2+\beta z_1^2+\gamma z_2^2+rz_1z_2.$$

Using $\beta=\gamma=0$ yields (6). □

Corollary 3.2. *Any two matrix triples satisfying* (5) *and sharing the same seven invariants*

$$(a,b,c,\alpha,p,q,r)$$

have exactly the same characteristic quasipolynomial for every pair of delays.

At zero delay,

$$\Delta(\lambda;0,0)=\det\left[\lambda I-(A+B+C)\right].$$

Hence

$$A+B+C\ \text{Hurwitz}\Leftrightarrow a+b+c<0,\quad \alpha+p+q+r>0. \tag{7}$$

4. Exact Elimination of the Delay Phases

Let $\lambda=i\omega$, $\omega>0$, and write

$$z_j=\mathrm{e}^{-i\omega\tau_j},\ |z_j|=1.$$

Define

$$d_0=\alpha-\omega^2-ia\omega,\ d_1=p-ib\omega,\ d_2=q-ic\omega.$$

Then (6) becomes

$$d_0+d_1z_1+d_2z_2+rz_1z_2=0. \tag{8}$$

Regrouping,

$$(d_2+rz_1)z_2=-(d_0+d_1z_1). \tag{9}$$

Lemma 4.1 (Exact phase elimination). *For fixed* $\omega>0$, *there exists*

$(z_1, z_2) \in \mathbb{T}^2$ *satisfying* (8) *if and only if*

$$|K(\omega)| \le 2|H(\omega)|, \tag{10}$$

where

$$K = |d_0|^2 + |d_1|^2 - |d_2|^2 - r^2$$

and

$$H = d_1\overline{d_0} - r\overline{d_2}.$$

Proof. Fix $z_1 \in \mathbb{T}$ and set

$$N = d_0 + d_1 z_1, \quad D = d_2 + r z_1.$$

Equation (9) is $Dz_2 = -N$. If $D \neq 0$, a unit-modulus solution exists if and only if

$$|N| = |D|,$$

in which case $z_2 = -N/D$ and automatically $|z_2| = 1$. If $D = 0$, solvability requires $N = 0$; when this holds, every $z_2 \in \mathbb{T}$ is admissible. Thus, including the singular case, existence of a unit-modulus z_2 is *exactly* equivalent to

$$|d_0 + d_1 z_1| = |d_2 + r z_1|.$$

No division by D is used to derive this condition, so the elimination neither discards singular solutions nor introduces spurious ones.

Squaring and expanding gives

$$K + 2\Re(Hz_1) = 0.$$

As z_1 runs over $\mathbb{T}$, $\Re(Hz_1)$ runs over $[-|H|, |H|]$. Hence a solution exists if and only if (10) holds. □

Remark 4.2 (Relation to singular crossing-set elimination). For quasipolynomials containing a mixed term $e^{-\lambda(\tau_1+\tau_2)}$, elimination procedures may create spurious crossing sets at singular frequencies if a vanishing factor is divided out. This issue is emphasized by Naghnaeian and Gu [10]. The modulus equivalence above is established before any quotient is formed and explicitly retains the case $D = N = 0$; consequently, the crossing criterion below is an equivalence on the original characteristic equation rather than on an associated equation obtained by division.

Because τ_1, τ_2 vary independently, every pair in $\mathbb{T}^2$ is realizable for fixed $\omega > 0$. If $z_j = e^{i\psi_j}$, then

$$\tau_j = \frac{-\psi_j + 2\pi k_j}{\omega}$$

is nonnegative for sufficiently large integers k_j.

5. The Invariant Quartic

Set

$$y = \omega^2.$$

Direct calculation yields

$$K(y) = y^2 + \left(a^2 + b^2 - c^2 - 2\alpha\right)y + \alpha^2 + p^2 - q^2 - r^2, \tag{11}$$

and

$$\left|H(y)\right|^2 = \left[p\alpha - rq + (ab - p)y\right]^2 + y\left[pa - b\alpha - rc + by\right]^2. \tag{12}$$

Define

$$\boxed{\begin{aligned} F(y) = &\left[y^2 + \left(a^2 + b^2 - c^2 - 2\alpha\right)y + \alpha^2 + p^2 - q^2 - r^2\right]^2 \\ &- 4\left[p\alpha - rq + (ab - p)y\right]^2 - 4y\left[pa - b\alpha - rc + by\right]^2. \end{aligned}} \tag{13}$$

Theorem 5.1 (Exact crossing criterion). *For every* $\omega > 0$,

$$\boxed{\exists \tau_1, \tau_2 \geq 0 : \Delta(i\omega; \tau_1, \tau_2) = 0 \Leftrightarrow F\left(\omega^2\right) \leq 0.} \tag{14}$$

Proof. The phase-elimination lemma gives

$$\left|K\right| \leq 2\left|H\right| \Leftrightarrow K^2 - 4\left|H\right|^2 \leq 0.$$

By definition the latter quantity is $F\left(\omega^2\right)$. Independent delays realize every admissible phase pair. □

Corollary 5.2 (Exact realizable crossing-frequency set). *Define*

$$\Omega_{\text{cr}} = \left\{\omega > 0 : \exists (\tau_1, \tau_2) \in [0, \infty)^2 \text{ such that } \Delta(i\omega; \tau_1, \tau_2) = 0\right\}.$$

Then

$$\boxed{\Omega_{\text{cr}} = \left\{\omega > 0 : F\left(\omega^2\right) \leq 0\right\}.}$$

Equivalently,

$$\mathcal{Y}_{\text{cr}} = \left\{y > 0 : F(y) \leq 0\right\}.$$

Thus, positive zeros of F are not, in general, all crossing frequencies. Generically they form endpoints of crossing-frequency bands, while $F(y) < 0$ *describes their interiors.*

Corollary 5.3 (Number of connected crossing-frequency components). *The set* Ω_{cr} *has at most two connected components. Each component is either a nondegenerate interval (possibly accumulating at* 0^+*) or, in a tangency case, a singleton.*

Proof. The polynomial *F* is a real monic quartic and tends to $+\infty$ as $y \to +\infty$. At most one component of $\left\{y > 0 : F(y) \leq 0\right\}$ can accumulate at 0^+; because $F(y) \to +\infty$, such a component must terminate at a positive zero and therefore uses at least one positive zero counted with multiplicity. Every other nondegenerate component is bounded away from the origin and requires two boundary zeros, while an isolated tangency component requires an even-multiplicity zero of multiplicity at least two. Hence three connected components would require at least $1 + 2 + 2 = 5$ zero multiplicities if one component accumulates at 0^+, or at least $2 + 2 + 2 = 6$ otherwise. Both are impossible for a quartic. Thus, there are

at most two connected components. The homeomorphism $y \mapsto \sqrt{y}$ preserves their number and type. □

Remark 5.4. If $F(y)<0$ and $H(y)\neq 0$, the phase equation has two distinct solutions for z_1. If $F(y)=0$ and $H(y)\neq 0$, these branches merge. The degenerate case $F(y)=H(y)=0$ must be treated separately.

The constant coefficient has a useful structural factorization.

Proposition 5.5. *Under* (5),

$$\boxed{F(0)=\prod_{\varepsilon_1,\varepsilon_2=\pm 1}\det\left(A+\varepsilon_1 B+\varepsilon_2 C\right).} \tag{15}$$

Proof. Since

$$\det\left(A+sB+tC\right)=\alpha+sp+tq+str$$

for $s,t\in\{\pm 1\}$, multiplying the four signed determinants gives

$$(\alpha+p+q+r)(\alpha+p-q-r)(\alpha-p+q-r)(\alpha-p-q+r),$$

which is exactly $F(0)$.

Proposition 5.6 (Zero frequency is not a free torus phase). *For every finite* (τ_1,τ_2),

$$\mathrm{e}^{-0\tau_1}=\mathrm{e}^{-0\tau_2}=1$$

and hence

$$\Delta(0;\tau_1,\tau_2)=\det\left(A+B+C\right)=\alpha+p+q+r.$$

Therefore, a finite-delay zero characteristic root exists if and only if

$$\det\left(A+B+C\right)=0.$$

In particular, $F(0)=0$ *does not by itself imply a finite-delay zero root.*

Proof. The first identity is immediate at $\lambda=0$. Substitution in the characteristic determinant gives the stated formula. The signed determinants in (15) arise from the algebraic limit of the phase-elimination polynomial at $y=0$; except for the $+,+$ configuration, they are not physically realizable zero-frequency phase pairs for finite delays. □

6. Exact Delay-Independent Stability Criterion

Lemma 6.1 (Finite-path spectral continuation). *Let the delay pair vary continuously on a compact path contained in* $[0,\infty)^2$. *For the retarded system* (1), *exponential stability is locally robust along that path, and a change in the number of characteristic roots in the open right half-plane can occur only when a characteristic root reaches the imaginary axis.*

Proof. This is a standard consequence of the spectral theory of retarded functional differential equations: the spectral limit depends continuously on the delay data under the relevant perturbations, and only finitely many characteristic roots lie to the right of any prescribed vertical line. For the zero-delay limit in particular, see Michiels and Niculescu ([12], Theorems 1.15-1.16, pp. 12-13): their spectral-abscissa continuity statement explicitly includes points where the retarded system

reduces to an ordinary differential system, and their next theorem links stability loss to imaginary-axis roots. See Cooke and Ferreira [2] and the general treatment in Hale and Verduyn Lunel [13]. The compactness of the finite-delay path excludes any need for an infinite-delay limiting argument.

Theorem 6.2 (Main DIS theorem). *Under* (5), *system* (1) *is asymptotically stable for every finite pair of independent delays if and only if*

$$a+b+c<0, \quad \alpha+p+q+r>0, \tag{16}$$

and

$$\boxed{F(y)>0 \quad \forall y>0.} \tag{17}$$

Proof. Necessity follows first by setting $\tau_1=\tau_2=0$, which yields (16). If $F(y_*)\le 0$ for some $y_*>0$, Theorem 5.1 produces finite delays with an imaginary root, contradicting asymptotic stability.

For sufficiency, fix an arbitrary finite pair (τ_1^*,τ_2^*). Consider

$$(\tau_1(s),\tau_2(s))=s(\tau_1^*,\tau_2^*), \quad 0\le s\le 1.$$

At $s=0$, the matrix $A+B+C$ is Hurwitz. By Lemma 6.1, exponential stability persists for all sufficiently small positive s.

Assume, for contradiction, that stability is lost somewhere on the compact path $s\in[0,1]$. Lemma 6.1 then implies that at a first loss of stability a characteristic root lies on the imaginary axis. A zero root is impossible by Proposition 5.6 and (16). A nonzero imaginary root is impossible by (17) and Theorem 5.1. Hence no stability loss can occur on the path, and the target pair (τ_1^*,τ_2^*) is stable. Since the finite target pair was arbitrary, the system is delay-independently stable. □

Remark 6.3. No separate assumption that A is Hurwitz is needed in the theorem. The proof concerns every fixed finite target pair of delays and never requires extending the continuation path to infinite delay.

Remark 6.4 (The limiting case $F(0)=0$). The strict positivity condition is imposed only for $y>0$. Proposition 5.6 shows that the torus interpretation used for $\omega>0$ cannot be extended literally to $\omega=0$. Consequently, $F(0)=0$ may coexist with ordinary finite-delay delay-independent stability provided $A+B+C$ is Hurwitz and $F(y)>0$ for every $y>0$. Such equality may indicate loss of a uniform margin in an infinite-delay limit, but it is not a finite-delay zero-root condition.

Exact Finite Positivity Test

Write

$$F(y)=y^4+A_3y^3+A_2y^2+A_1y+A_0,$$

so that

$$F'(y)=4y^3+3A_3y^2+2A_2y+A_1.$$

Proposition 6.5 (Exact quartic positivity test). *The condition*

$$F(y)>0 \quad \forall y>0$$

is equivalent to the conjunction of the following two finite conditions:

1) F *is positive immediately to the right of the origin, namely either*

$$A_0 > 0,$$

or $A_0 = 0$ *and the first nonzero coefficient in the ordered list*

$$A_1, A_2, A_3, 1$$

is positive;

2) *for every positive real root* ξ *of* F',

$$F(\xi) > 0.$$

Since F' *is cubic, at most three positive stationary points need to be checked. Equivalently, after classifying stationary points, only positive local minima are decisive, a quartic has at most two of them.*

Proof. Necessity is immediate. Conversely, the first condition guarantees $F(y) > 0$ on some interval $(0, \varepsilon)$. Since F is monic,

$$F(y) \to +\infty \quad (y \to +\infty).$$

If $F(y_0) \le 0$ for some $y_0 > 0$, continuity together with positivity near 0^+ and for sufficiently large y implies that F attains a nonpositive minimum at some interior point $\xi > 0$. Hence $F'(\xi) = 0$ and $F(\xi) \le 0$, contradicting the second condition. □

Remark 6.6. Testing all positive real roots of the cubic F' avoids any ambiguity caused by degenerate stationary points with $F'' = 0$. The local-minimum formulation is equivalent, but the all-stationary-point version is simpler to implement robustly.

In particular, $A_0 < 0$ immediately rules out positivity on $(0, \infty)$. When $A_0 = 0$, the first-nonzero-coefficient rule handles all possible orders of contact with the origin, including $F(y) = y^4$.

A simple sufficient condition is

$$A_0 > 0, \quad A_1, A_2, A_3 \ge 0.$$

7. Vanishing Mixed Interaction: Triangle Geometry

Suppose

$$r = 0.$$

Then

$$\Delta(\lambda) = P_0(\lambda) + P_1(\lambda)\mathrm{e}^{-\lambda\tau_1} + P_2(\lambda)\mathrm{e}^{-\lambda\tau_2},$$

where

$$P_0(\lambda) = \lambda^2 - a\lambda + \alpha, \ P_1(\lambda) = p - b\lambda, \ P_2(\lambda) = q - c\lambda.$$

For $y = \omega^2$, define

$$L_0^2(y) = y^2 + (a^2 - 2\alpha)y + \alpha^2,$$

$$L_1^2(y) = p^2 + b^2 y, \quad L_2^2(y) = q^2 + c^2 y.$$

Theorem 7.1. *For* $r=0$, *a positive frequency* $\omega=\sqrt{y}$ *can be an imaginary characteristic root for some delays if and only if*

$$\boxed{\left|L_1(y)-L_2(y)\right|\le L_0(y)\le L_1(y)+L_2(y).} \tag{18}$$

Proof. At $\lambda=i\omega$,

$$P_0+P_1z_1+P_2z_2=0.$$

The independent phases rotate the complex numbers P_1 and P_2 freely without changing their lengths. Hence the equation is solvable exactly when three vectors of lengths L_0,L_1,L_2 close to form a triangle. □

Direct algebra gives

$$F(y)=\left[L_0^2-\left(L_1+L_2\right)^2\right]\left[L_0^2-\left(L_1-L_2\right)^2\right]. \tag{19}$$

Whenever the triangle exists,

$$-F(y)=16\mathcal{A}(y)^2,$$

where $\mathcal{A}$ is its area.

Since $L_0(y)\sim y$ while $L_1,L_2=O\left(\sqrt{y}\right)$, the high-frequency non-crossing branch is $L_0>L_1+L_2$. Thus, under zero-delay stability,

$$\text{DIS}\Leftrightarrow L_0(y)>L_1(y)+L_2(y)\quad\forall y>0.$$

8. Critical Delay Reconstruction and Algebraic Boundaries

Suppose $F(y)\le 0$ for some $y>0$, and set $\omega=\sqrt{y}$. Recall that admissible phases satisfy

$$K(y)+2\,\mathrm{Re}\left(H(y)z_1\right)=0,\quad |z_1|=1. \tag{20}$$

Proposition 8.1 (Complete reconstruction of critical phases). *Let* $y>0$ *satisfy* $F(y)\le 0$, *and let* $\omega=\sqrt{y}$.

If $H(y)\neq 0$, *write*

$$H(y)=|H(y)|\mathrm{e}^{i\phi},\quad z_1=\mathrm{e}^{i\psi_1}.$$

Then all admissible first phases are

$$\boxed{\psi_1^{\pm}=-\phi\pm\arccos\left(-\frac{K(y)}{2|H(y)|}\right)\ (\mathrm{mod}\,2\pi).} \tag{21}$$

For each such z_1, *define*

$$N=d_0+d_1z_1,\quad D=d_2+rz_1.$$

Equation (20) *is equivalent to* $|N|=|D|$. *Hence*:

1) *if* $D\neq 0$, *then*

$$\boxed{z_2=-\frac{N}{D},} \tag{22}$$

and automatically $|z_2|=1$;

2) *if* $D=0$, *then* $|N|=|D|$ *forces* $N=0$, *and every* $z_2\in\mathbb{T}$ *is admissible.*

If $H(y)=0$, *then*

$$F(y)=K(y)^2.$$

Since $F(y)\le 0$, *necessarily*

$$F(y)=0,\quad K(y)=0.$$

Therefore (20) *holds for every* $z_1\in\mathbb{T}$. *For each such* z_1, *one again has* $|N|=|D|$: *if* $D\neq 0$, *the unique admissible second phase is* $z_2=-N/D$; *if* $D=0$, *then* $N=0$ *and every* $z_2\in\mathbb{T}$ *is admissible.*

Remark 8.2. Thus, no admissible solution is lost through the quotient in (22). A vanishing denominator can occur only together with a vanishing numerator once z_1 satisfies the phase condition. The apparently singular case therefore produces a continuum of admissible second phases rather than an obstruction.

If $z_j=\mathrm{e}^{i\psi_j}$, every admissible phase pair is realized by finite nonnegative delays

$$\boxed{\tau_j^{(k)}=\frac{-\psi_j+2\pi k}{\omega},} \tag{23}$$

where integers k are chosen sufficiently large that $\tau_j^{(k)}\ge 0$. Hence the reconstruction is complete for every $\omega>0$.

The reconstructed critical-delay families locate parameter values at which the characteristic equation has imaginary roots; identifying the stable side of these families also requires checking root multiplicity and the direction of crossing.

Generic parameter-space boundaries occur when a positive local minimum of F touches zero:

$$F(y_*;\mu_*)=0,\quad F_y(y_*;\mu_*)=0,\quad y_*>0. \tag{24}$$

Equivalently,

$$\mathrm{Res}_y\left(F,F_y\right)=0,$$

subject to the positive-real-root and zero-delay stability constraints.

Parameter boundaries must also account for loss of the zero-delay Hurwitz conditions. With $\alpha+p+q+r>0$, the equality $a+b+c=0$ gives an imaginary pair at zero delays; the equality $\alpha+p+q+r=0$ gives a zero characteristic root for every finite delay pair. A separate endpoint mechanism concerns the sign of $F(y)$ as $y\downarrow 0$. Here $F(0)=0$ alone does not imply finite-delay stability loss: the sign for small positive y, determined by the first nonzero term of the quartic at the origin, must still be checked. These endpoint mechanisms complement the interior positive-frequency tangency $F(y_*)=F_y(y_*)=0$ with $y_*>0$.

9. Hermitian Representation

Define

$$\mathcal{Q}(y)=\begin{pmatrix} K(y) & 2H(y) \\ \overline{2H(y)} & K(y)\end{pmatrix}. \tag{25}$$

Its eigenvalues are

$$K(y)\pm 2|H(y)|,$$

and

$$\det \mathcal{Q}(y) = F(y).$$

Proposition 9.1 (Global Hermitian characterization). *For a fixed* $y > 0$,

$$F(y) > 0$$

implies that $\mathcal{Q}(y)$ *is definite, but it may be either positive or negative definite. More precisely,*

$$\mathcal{Q}(y) \succ 0 \Leftrightarrow K(y) > 2|H(y)|,$$

whereas

$$\mathcal{Q}(y) \prec 0 \Leftrightarrow K(y) < -2|H(y)|.$$

On the complete positive frequency axis,

$$\boxed{F(y) > 0 \quad \forall y > 0 \Leftrightarrow \mathcal{Q}(y) \succ 0 \quad \forall y > 0.} \tag{26}$$

Proof. The pointwise statements follow from the eigenvalues

$$K(y) \pm 2|H(y)|.$$

For the global implication, assume $F(y) > 0$ for all $y > 0$. Since

$$K(y) = y^2 + O(y),$$

we have $K(y) > 0$ for all sufficiently large y. If K were nonpositive at some positive point, continuity would produce $y_0 > 0$ with $K(y_0) = 0$. But then

$$F(y_0) = -4|H(y_0)|^2 \le 0,$$

a contradiction. Hence $K(y) > 0$ for every $y > 0$. Combining $K > 0$ with

$$F = K^2 - 4|H|^2 > 0$$

gives $K > 2|H|$, so both eigenvalues of $\mathcal{Q}$ are positive. The converse in (26) follows immediately from

$$F(y) = \det \mathcal{Q}(y) > 0.$$

□

Remark 9.2. The global quantifier in (26) is essential: pointwise positivity of $F(y)$ alone does not imply $\mathcal{Q}(y) \succ 0$.

Thus, the main theorem may equivalently be expressed as

$$A + B + C \text{ Hurwitz}, \quad \mathcal{Q}(y) \succ 0 \quad \forall y > 0.$$

10. Simultaneously Triangularizable Systems

Assume A, B, C are simultaneously triangularizable over $\mathbb{R}$:

$$S^{-1}AS = \begin{pmatrix} a_1 & * \\ 0 & a_2 \end{pmatrix}, \quad S^{-1}BS = \begin{pmatrix} b_1 & * \\ 0 & b_2 \end{pmatrix}, \quad S^{-1}CS = \begin{pmatrix} c_1 & * \\ 0 & c_2 \end{pmatrix}.$$

Then

$$\Delta(\lambda) = \Delta_1(\lambda)\Delta_2(\lambda),$$

where

$$\Delta_j(\lambda) = \lambda - a_j - b_j e^{-\lambda\tau_1} - c_j e^{-\lambda\tau_2}.$$

For the scalar equation

$$\dot{x}(t) = ax(t) + bx(t-\tau_1) + cx(t-\tau_2),$$

independent-delay asymptotic stability is equivalent to

$$a+b+c<0, \quad |a| \geq |b| + |c|.$$

Therefore, the simultaneously triangularizable 2×2 system is DIS if and only if, for $j=1,2$,

$$a_j + b_j + c_j < 0, \quad |a_j| \geq |b_j| + |c_j|.$$

11. Numerical Illustrations

11.1. Delay-Independent Example

Let

$$A = \begin{pmatrix} -3 & 0 \\ 0 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

The invariants are

$$(a,b,c,\alpha,p,q,r) = \left(-7, \frac{1}{2}, \frac{1}{2}, 12, -2, -\frac{3}{2}, \frac{1}{4}\right).$$

Moreover,

$$A+B+C = \operatorname{diag}\left(-\frac{5}{2}, -\frac{7}{2}\right)$$

is Hurwitz and

$$F(y) = \frac{(4y+35)^2(4y+63)^2}{256} > 0 \quad (y>0).$$

Hence the system is delay-independently stable.

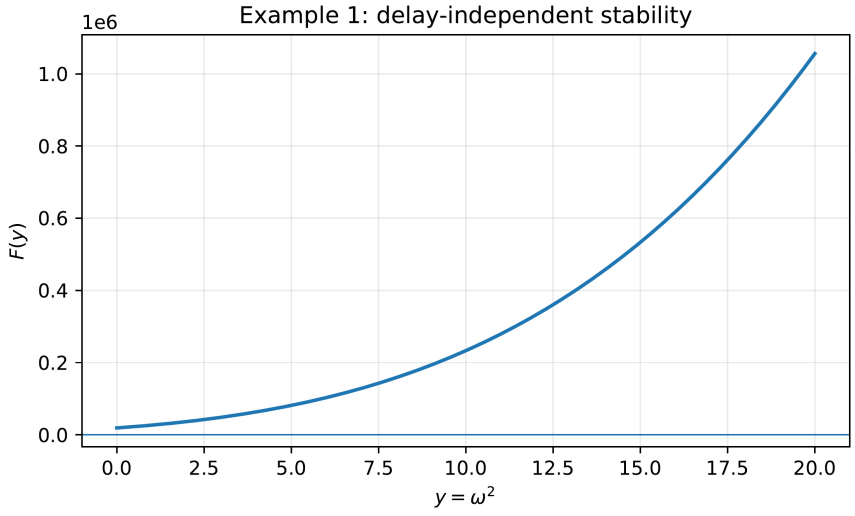


Figure 1. Invariant quartic for Example 1. The polynomial remains strictly positive on the positive frequency axis.

Figure 1 shows the quartic remaining positive for every positive squared frequency.

11.2. Zero-Delay Stable but Delay-Destabilized

Let

$$A=\begin{pmatrix}-1 & 0\\ 0 & -2\end{pmatrix},\quad B=\begin{pmatrix}-1 & -1\\ 0 & 0\end{pmatrix},\quad C=\begin{pmatrix}0 & 0\\ -1 & -1\end{pmatrix}.$$

Then

$$(a,b,c,\alpha,p,q,r)=(-3,-1,-1,2,2,1,0),$$

and $A+B+C$ is Hurwitz. However,

$$F(y)=y^4+6y^3+3y^2-26y-15,$$

so

$$F(1)=-31<0.$$

Thus $\omega=1$ is an admissible crossing frequency.

One critical pair is approximately

$$(\tau_1,\tau_2)=(1.951541,3.349872),$$

and a second fundamental branch gives approximately

$$(\tau_1,\tau_2)=(2.760848,2.006018),$$

up to the periodic replicas generated by $2\pi/\omega$.

Since $r=0$,

$$L_0=\sqrt{10},\quad L_1=\sqrt{5},\quad L_2=\sqrt{2},$$

which satisfy the triangle inequalities.

Figure 2 and Figure 3 show, respectively, the negative frequency interval and the reconstructed critical-delay branches.

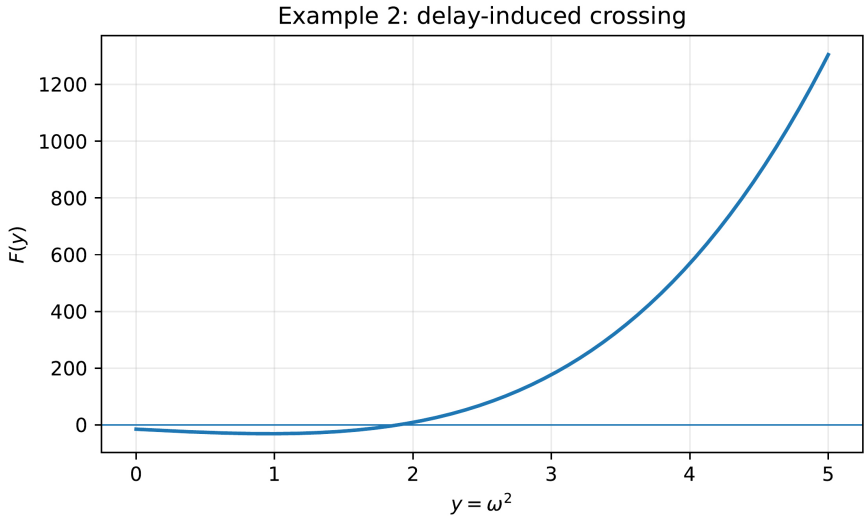


Figure 2. Invariant quartic for Example 2. The negative interval identifies frequencies that can be realized as imaginary-axis crossings by suitable independent delays.

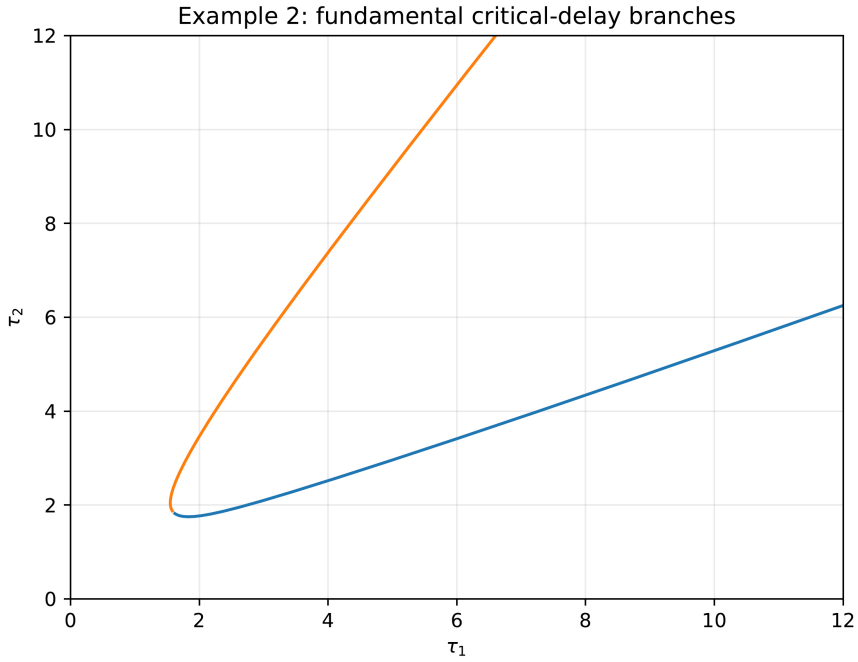


Figure 3. Two fundamental critical-delay branches reconstructed from the admissible crossing-frequency band of Example 2. Integer phase replicas generate the remaining crossing curves.

11.3. A Limiting $F(0)=0$ Case That Remains DIS

To illustrate why the condition in the main theorem is imposed on $y>0$, consider

$$A=\begin{pmatrix}-3 & 0\\ 0 & -4\end{pmatrix},\quad B=\begin{pmatrix}1/2 & 0\\ 0 & 0\end{pmatrix},\quad C=\begin{pmatrix}0 & 0\\ 0 & -4\end{pmatrix}.$$

Both delayed matrices have rank at most one, while

$$A+B+C=\operatorname{diag}\left(-\frac{5}{2},-8\right)$$

is Hurwitz. The invariants are

$$(a,b,c,\alpha,p,q,r)=\left(-7,\frac{1}{2},-4,12,-2,12,-2\right),$$

and direct substitution gives

$$F(y)=y^2\left(y^2+\frac{35}{2}y+\frac{1225}{16}\right).$$

Hence

$$F(0)=0,\quad F(y)>0\quad \forall y>0.$$

The main theorem therefore classifies this system as delay-independently stable for every finite delay pair. This example demonstrates concretely that replacing the condition $F(y)>0$ for $y>0$ by $F(y)>0$ for $y\geq 0$ would be unnecessarily restrictive.

11.4. Tangency Boundary

Consider

$$A=\begin{pmatrix}-4 & 0\\ 3 & -4\end{pmatrix},$$

$$B(\mu)=\mu\begin{pmatrix}-2 & 1\\ 2 & -1\end{pmatrix},\quad C(\mu)=\mu\begin{pmatrix}0 & -4\\ 0 & 4\end{pmatrix}.$$

The invariants are

$$a=-8,\ b=-3\mu,\ c=4\mu,\ \alpha=16,\ p=9\mu,\ q=-4\mu,\ r=0.$$

The generic boundary equations

$$F(y,\mu)=0,\quad F_y(y,\mu)=0$$

possess the positive solution

$$\mu_*\approx 0.9847484936,\quad y_*\approx 8.3213406762,$$

hence

$$\omega_*\approx 2.884673409.$$

At this parameter value the quartic is tangent to the horizontal axis at the positive local minimum y_*. Figure 4 displays this tangency.

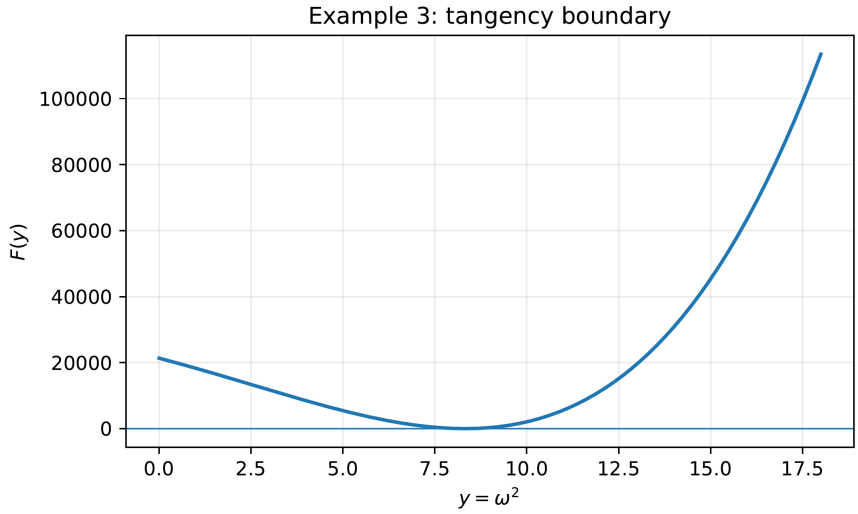


Figure 4. Invariant quartic at the boundary parameter of Example 3. The positive local minimum is tangent to the horizontal axis.

12. Relation to Previous Work

Proposition 12.1 (Embedding in the standard two-delay cross-term form). *Under* $\det B=\det C=0$, *the characteristic quasipolynomial belongs to the standard two-delay cross-term family*

$$P_0(\lambda)+P_1(\lambda)\mathrm{e}^{-\lambda\tau_1}+P_2(\lambda)\mathrm{e}^{-\lambda\tau_2}+P_{12}(\lambda)\mathrm{e}^{-\lambda(\tau_1+\tau_2)},$$

with the explicit identification

$$P_0(\lambda)=\lambda^2-a\lambda+\alpha,\quad P_1(\lambda)=p-b\lambda,$$
$$P_2(\lambda)=q-c\lambda,\quad P_{12}(\lambda)=r.$$

Thus, the present class is a dimension-two invariant specialization of the rank-one/two-channel quasipolynomial structures treated in the previous literature.

Proof. The statement follows directly from the invariant characteristic representation (6). In dimension two, rank-one delayed matrices remove the pure quadratic phase terms $e^{-2\lambda\tau_1}$ and $e^{-2\lambda\tau_2}$, while the mixed invariant $r=\mathrm{tr}(\mathrm{adj}(B)C)$ remains as the coefficient of the cross term. □

The general idea of replacing transcendental all-delay stability questions by algebraic or frequency-domain tests predates the present work. Feng, Liu and Zhao [4] gave necessary and sufficient algebraic conditions for multidelay systems, and Chen and Latchman [5] developed necessary and sufficient frequency-sweeping tests. Yu, Zhong and Li [9] later gave another algebraic criterion for systems with two delays.

The closest prior result is Wu and Ren [6], who considered two-delay retarded systems with rank-one delayed matrices in general state dimension and obtained necessary and sufficient criteria through real roots of auxiliary polynomials. Proposition 12.1 makes the relationship explicit at the characteristic level. Accordingly, neither the rank-one setting, the two-delay setting, the cross-term quasipolynomial structure, nor the general existence of a polynomial DIS criterion is claimed here as new.

What is specialized here is not merely the state dimension in an entrywise formula. The 2×2 determinant identities compress the complete characteristic equation to the seven simultaneous-similarity invariants (a,b,c,α,p,q,r). The subsequent torus elimination closes directly on these invariants and yields the single scalar condition $F(y)\le 0$ for realizable crossings, together with the finite positivity test for DIS. This is the structural simplification on which the contribution rests.

Theorem 12.2 (Planar specialization of Wu and Ren's Theorem 3.1). *For the common subclass with* $\mathrm{rank}\,B=\mathrm{rank}\,C=1$, *let* $A_L(\omega)$ *denote the polynomial defined by Wu and Ren* [6, *Theorem* 3.1, Eq. (11), *pp.* 629-630]. *Under the coefficient identification of Proposition* 12.1,

$$A_L(\omega)=-F(\omega^2).$$

Their independent-delay criterion therefore becomes exactly

$$a+b+c<0,\quad \alpha+p+q+r>0,\quad F(y)>0\quad \textit{for every}\quad y>0.$$

Proof. To distinguish their auxiliary functions from our invariants, denote the functions in their Equations (7)-(10) by $\mathfrak{a},\cdots,\mathfrak{h}$. Substitution of $P_0(i\omega)=\alpha-\omega^2-ia\omega$, $P_1(i\omega)=p-ib\omega$, $P_2(i\omega)=q-ic\omega$, and $P_{12}(i\omega)=r$ gives

$$\mathfrak{a} = \alpha + r - \omega^2, \qquad \mathfrak{b} = a\omega,$$
$$\mathfrak{c} = -a\omega, \qquad \mathfrak{d} = \alpha - r - \omega^2,$$
$$\mathfrak{e} = p + q, \qquad \mathfrak{f} = (c-b)\omega,$$
$$\mathfrak{g} = -(b+c)\omega, \qquad \mathfrak{h} = q - p.$$

Inserting these eight expressions into their Equation (11) yields $A_L(\omega) = -F(\omega^2)$. Also,

$$\Delta(\lambda;0,0) = \lambda^2 - (a+b+c)\lambda + \alpha + p + q + r,$$

so their zero-delay Hurwitz condition gives the first two inequalities. Their remaining condition excludes every nonzero real root of A_L, equivalently every positive real root of F. Since F is monic and tends to $+\infty$, absence of positive roots is equivalent to $F(y) > 0$ for every $y > 0$: a negative value would force a positive root by continuity. This proves the stated specialization. □

Wu and Ren's Remark 1 ([6], p. 630) permits roots at $\omega = 0$, consistently with the admissible limiting case $F(0) = 0$ discussed here. Their planar example already produces an even polynomial of degree eight in ω ([6], Eq. (27), p. 635), hence a quartic in ω^2. The present formulation supplies its universal coefficients through seven simultaneous-similarity invariants, together with the direct phase-solvability proof that retains singular phases and the explicit critical-delay reconstruction. Wu and Ren assume nonzero rank-one delayed matrices; the direct proofs in Sections 4 and 6 also cover the zero-rank cases allowed in this paper, and the polynomial identity remains valid there.

The present contribution is therefore the explicit invariant specialization to state dimension two. Even-polynomial reductions in the frequency variable already occur in the rank-one two-delay literature, so quartic degree after the substitution $y = \omega^2$ is not claimed by itself as a novelty. Here the entire characteristic quasipolynomial is represented universally by seven simultaneous-similarity invariants; both independent phases are eliminated exactly; and the resulting quartic is obtained in a closed invariant form.

Gu, Niculescu and Chen [7] provide a detailed crossing-set and stability-crossing-curve framework for scalar quasipolynomials of arbitrary order with two delays. Their crossing-frequency set is described by inequality constraints and forms finitely many bounded intervals. The present result is consistent with that geometry but, for the rank-one 2×2 matrix class, compresses the crossing-frequency set to the single inequality $F(\omega^2) \le 0$.

Michiels and Niculescu [8] provide a general multidelay framework and distinguish independent from constrained delays. That distinction is essential here because the exact torus argument relies on independent phase coverage.

Naghnaeian and Gu [10] show that mixed exponential terms require careful elimination to avoid spurious crossing sets. The present derivation handles the singular case $d_2 + rz_1 = 0$ before division.

Alikoc and Ergenc [11] give a substantially more general polynomial method based on self-inversive auxiliary polynomials. The advantage of the present spe-

cialization is not generality but transparency: no phase sweep remains after the seven invariants have been computed.

Table 1 summarizes the comparison with representative exact approaches.

Table 1. Position of the present reduction relative to representative exact approaches.

Method	Scope	Type of result	Main computational object
Feng-Liu-Zhao	multidelay LTI systems	exact DIS	algebraic equations
Chen-Latchman	commensurate and noncommensurate LTI systems	exact DIS	frequency sweeping/structured singular values
Wu-Ren	two delays, rank-one delayed matrices, general state dimension	exact DIS	auxiliary real polynomials
Gu-Niculescu-Chen	scalar quasipolynomials of arbitrary order, two delays	exact crossing set	crossing-frequency inequalities and curves
Michiels-Niculescu	multiple delays, independent or constrained	exact DIS characterization	spectra of complex matrices
Naghnaeian-Gu	two scalar-delay channels with cross term	exact crossing set	geometric elimination with singular-case correction
Alikoc-Ergenc	general LTI systems with constant delays	exact DIS	self-inversive auxiliary polynomials
Present work	2×2, two independent rank-one channels	exact DIS and crossing set	seven invariants + one explicit quartic

13. Discussion

The reduction reveals structure that is hidden in a direct entrywise treatment. First, seven scalar simultaneous-similarity invariants encode the complete characteristic quasipolynomial in the rank-one 2×2 class. Second, the mixed invariant

$$r = \mathrm{tr}\left(\mathrm{adj}\left(B\right)C\right)$$

is exactly the coefficient of the mixed exponential $\mathrm{e}^{-\lambda\left(\tau_1+\tau_2\right)}$, giving it a direct characteristic-level interpretation.

Third, the quartic F carries several complementary meanings. Its sign determines whether a frequency is realizable as an imaginary characteristic root; its positive roots generically delimit crossing-frequency intervals; its local minima decide DIS; its discriminant describes generic parameter boundaries; and its constant term factors into four signed determinants.

The finite critical-point test is also numerically important. A coarse frequency sweep can miss a narrow interval where $F<0$, whereas checking the endpoint value $F(0)$ and all positive stationary points cannot miss such an interval in exact arithmetic. To guarantee this conclusion in a finite-precision implementation, all positive roots of F' must be isolated and the signs of F certified, using certified root isolation or interval arithmetic. This is particularly important near tangencies, where F and F' can be zero or close to zero; an unresolved interval sign must not be treated as a certificate of positivity.

The method has clear limitations. It is dimension-specific: for $n>2$, the determinant no longer closes in seven invariants. It is also rank-specific: if $\det B\neq 0$ or $\det C\neq 0$, the characteristic equation acquires z_1^2 and z_2^2 terms and the present one-phase elimination no longer yields the same quartic. Finally, it assumes independent delays. Constrained or commensurate delays trace lower-dimensional subsets of the torus and require additional compatibility conditions.

A rank-one delayed channel arises when a scalar delayed feedback signal enters through one input direction: its coefficient has the form $B=u_1v_1^{\mathrm{T}}$, and a second channel has $C=u_2v_2^{\mathrm{T}}$. The two fixed delay values are treated as independent parameters, as for two feedback or transmission channels whose delays can be chosen separately. At each positive frequency, this independence allows both phase factors to cover the full unit circle without a prescribed relation between the delays, which is the assumption used in the elimination above.

Natural extensions include the full-rank 2×2 problem, strong or uniform delay-independent stability, constrained-delay relations, and robust parameter optimization based on the algebraic boundary $\mathrm{Disc}(F)=0$.

14. Conclusions

For a two-dimensional linear retarded system with two independent delays and rank-one delayed channels, the delay-independent stability problem admits a compact invariant reduction. The characteristic quasipolynomial is completely determined by seven simultaneous-similarity invariants. Exact elimination of the delay phases reduces the existence of nonzero imaginary roots to the sign of a single monic quartic $F(y)$ in the squared frequency.

The system is asymptotically stable for every finite delay pair if and only if the zero-delay matrix $A+B+C$ is Hurwitz and

$$F(y)>0 \quad \forall y>0.$$

Because F' is cubic, the positivity condition can be tested through at most two positive local minima. The same quartic also yields a determinant factorization at zero frequency, a Hermitian positivity representation, a triangle/Heron interpretation when $r=0$, explicit critical-delay families, and algebraic parameter boundaries.

The main value of the result is therefore not a new general theory of multidelay stability, but an explicit invariant characterization of a nontrivial low-dimensional

subclass in which a transcendental two-phase problem becomes a transparent univariate quartic test.

Author Contributions

All authors contributed equally to conceptualization, methodology, formal analysis, validation, writing—original draft, and writing—review and editing.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix

A. Expanded Coefficients of the Invariant Quartic

Writing

$$F(y) = y^4 + A_3 y^3 + A_2 y^2 + A_1 y + A_0,$$

one obtains

$$\begin{aligned}
A_3 &= 2a^2 - 4\alpha - 2b^2 - 2c^2, \\
A_2 &= a^4 - 4a^2\alpha - 2a^2b^2 - 2a^2c^2 + 6\alpha^2 + 4\alpha b^2 + 4\alpha c^2 \\
&\quad + b^4 - 2b^2c^2 + c^4 + 8bcr - 2p^2 - 2q^2 - 2r^2, \\
A_1 &= 2a^2\alpha^2 - 2a^2p^2 - 2a^2q^2 - 2a^2r^2 + 8abqr + 8acpr \\
&\quad - 4\alpha^3 - 2\alpha^2b^2 - 2\alpha^2c^2 - 8\alpha bcr + 4\alpha p^2 + 4\alpha q^2 + 4\alpha r^2 \\
&\quad + 2b^2p^2 - 2b^2q^2 - 2b^2r^2 - 2c^2p^2 + 2c^2q^2 - 2c^2r^2 - 8pqr,
\end{aligned}$$

and

$$A_0 = (\alpha + p + q + r)(\alpha + p - q - r)(\alpha - p + q - r)(\alpha - p - q + r).$$

B. Cubic Stationary-Point Test

The derivative is

$$G(y) = F'(y) = 4y^3 + 3A_3 y^2 + 2A_2 y + A_1.$$

After the substitution $y = x - A_3/4$, the normalized derivative becomes a depressed cubic whose coefficients may be written

$$P = \frac{8A_2 - 3A_3^2}{16}, \quad Q = \frac{A_3^3 - 4A_2A_3 + 8A_1}{32}.$$

For a robust exact implementation, compute all real roots of G, retain every positive root, and evaluate F at each of them. Since G is cubic, at most three positive stationary points are tested. This formulation automatically covers degenerate stationary points for which $F'' = 0$. After classification, only local minima are decisive, and a quartic has at most two local minima.

The discriminant of the derivative cubic is

$$\mathrm{Disc}(G) = -4\left(108A_1^2 - 108A_1A_2A_3 + 27A_1A_3^3 + 32A_2^3 - 9A_2^2A_3^2\right).$$

Thus $\mathrm{Disc}(G) = 0$ identifies multiple stationary points.

For completeness, the first nontrivial polynomial in the Sturm sequence of F and F', up to the conventional positive scaling, is

$$S_2(y) = \frac{3A_3^2 - 8A_2}{16} y^2 + \frac{2A_2A_3 - 12A_1}{16} y + \frac{A_1A_3 - 16A_0}{16}.$$

These identities provide an entirely algebraic route to root counting when a certified symbolic implementation is preferred over numerical cubic root finding.