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A Trapezoidal Fuzzy Heston Model Calibrated to Copper Futures Prices

Kankolongo Kadilu Patient[1*], Kumwimba Seya Didier[1], Panga Lutanda Grégoire[1], Balowayi Bondu Bernard[1], Mwania Wakosia José[2]

[1]Department of Mathematics and Computer Sciences, Université de Lubumbashi, Lubumbashi, Democratic Republic of the Congo
[2]Department of Economy and Management Sciences, Université de Lubumbashi, Lubumbashi, Democratic Republic of the Congo
Email: *Kankolongo.Kadilu@unilu.ac.cd, seyak@unilu.ac.cd, greglutandapanga@gmail.com, balowayibernard@gmail.com, josemwania@gmail.com

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Abstract

This paper develops a fuzzy stochastic volatility framework for pricing and calibrating copper futures contracts under parameter uncertainty. The classical Heston model is extended by representing its structural parameters as trapezoidal fuzzy numbers, allowing the model to account for epistemic uncertainty arising from limited data, market illiquidity, and structural misspecification. Using α-cut decomposition, the fuzzy pricing problem is reduced to a family of deterministic Heston-type models indexed by the confidence level $\alpha \in [0,1]$. A rigorous theoretical foundation is established by proving the existence, uniqueness, and probabilistic representation of the resulting α-level partial differential equations via an adapted Feynman-Kac theorem. To overcome the limitations of grid-based numerical solvers, a Fuzzy Deep Galerkin Method (FDGM) is proposed for solving the α-level PDEs and calibrating the model directly to market data. The methodology is applied to copper futures prices, and numerical results demonstrate that the fuzzy Heston model significantly outperforms classical benchmark models in terms of calibration accuracy and robustness. The proposed framework provides a flexible and computationally efficient tool for uncertainty-aware pricing in commodity markets.

Keywords

Heston Model, Trapezoidal Fuzzy Numbers, α-Cuts, Fuzzy Stochastic Volatility, Copper Futures, Fuzzy Deep Galerkin Method, Fuzzy Calibration, Uncertainty Quantification, Commodity Derivatives

1. Introduction

Commodity futures markets exhibit high volatility, structural uncertainty, and regime shifts influenced by macroeconomic factors, geopolitical events, and supply-demand imbalances [1] [2]. Copper, in particular, plays a central role in industrial production and economic activity, making accurate pricing of copper futures contracts crucial for hedging, investment, and risk management purposes.

Traditional stochastic volatility models, such as the Heston model [3], have been extensively applied to financial markets due to their ability to capture volatility smiles and term structure effects. Extensions to futures pricing rely on the martingale property of futures under the risk-neutral measure [4] [5]. Nevertheless, classical calibration techniques typically assume precise, point-valued parameters, which may fail to account for uncertainty in markets characterized by sparse or noisy data [6]. Ignoring such parameter uncertainty can lead to unstable pricing, overconfident risk measures, and poor out-of-sample performance.

To address these limitations, recent studies advocate incorporating robust, Bayesian, or interval-based methods in stochastic volatility models [7]. An alternative and complementary framework is provided by fuzzy set theory [8], which allows model parameters to be represented as fuzzy numbers, capturing epistemic uncertainty in a transparent and interpretable manner. By leveraging the α-cut representation, fuzzy parameters can be propagated through pricing models, producing confidence-dependent price bands rather than single-point estimates.

In the early years of the twenty-first century, a parallel strand of research has incorporated fuzzy uncertainty into option pricing by substituting crisp model inputs with fuzzy numbers and propagating imprecision through the extension principle and α-cut techniques. Within the Black-Scholes framework, seminal contributions include [9]-[15], who investigate fuzzified inputs such as the underlying price, volatility, and interest rate, as well as related numerical methods and defuzzification procedures.

A comprehensive review by [16] classifies the literature on fuzzy option pricing into four main approaches: 1) the predominant fuzzy-random methodology, which embeds fuzzy parameters into classical pricing formulas; 2) models that modify risk-neutral probabilities using fuzzy measures; 3) frameworks incorporating fuzzy payoffs, particularly in real option analysis; and 4) soft-computing techniques, including fuzzy neural networks. From an empirical perspective, [16] demonstrates that applying the Black-Scholes formula with triangular fuzzy inputs generally yields non-triangular fuzzy option prices. The study further evaluates three triangular post-approximation schemes on IBEX-35 options, showing that their accuracy varies with option moneyness and time to maturity.

However, [17] introduce a sophisticated stochastic model that captures complex market dynamics, such as mean-reverting convenience yields and time-varying volatility, which are critical for the accurate valuation of energy derivatives. By leveraging deep neural networks through the Deep Galerkin Method on crude oil futures, the authors provide a scalable solution for high-dimensional financial modeling that maintains high precision while reducing computational costs compared to grid-based methods. The Deep Galerkin Method (DGM), as introduced by [18], represents a significant shift in solving partial differential equations (PDEs) by using deep neural networks to approximate their solutions. Instead of relying on the traditional fixed-grid or mesh-based structures typical of methods like finite differences, DGM is a mesh-free approach that trains a network to minimize a loss function derived from the PDE's differential operator and its associated boundary or initial conditions. One of the primary advantages of this method is its ability to mitigate the "curse of dimensionality," making it particularly effective for high-dimensional PDEs where classical numerical techniques often fail or become computationally prohibitive. Furthermore, the framework has been broadened by [19] and [20] to solve complex problems in mean field games and optimal stochastic control. In the context of energy markets, this method has been successfully applied to accurately price European call options on crude oil futures within complex three-factor stochastic frameworks.

In this paper, we propose a fuzzy extension of the Heston model for copper futures pricing, in which key parameters, the mean-reversion speed, long-term variance, volatility of variance, initial variance, and correlation are modeled as trapezoidal fuzzy numbers. Using the Fuzzy Deep Galerkin Method (FDGM), we solve the corresponding α-level PDEs efficiently via deep neural networks, enabling the calibration of the fuzzy Heston model directly to observed copper futures prices. Our approach is consistent with the extension principle of fuzzy set theory [8] [21] and follows the methodology outlined by [22] for incorporating parameter uncertainty in derivative pricing. The main contribution of this paper is to introduce a Fuzzy Heston model for copper futures pricing and a Fuzzy Deep Galerkin Method for its numerical solution and calibration. By combining fuzzy set theory, stochastic volatility modeling, and deep learning-based PDE solvers, we provide a rigorous and computationally efficient framework that captures parameter uncertainty and improves empirical pricing performance.

The remainder of this paper is structured as follows. Section 2 recalls the classical Heston stochastic volatility model applied to futures pricing. Section 3 reviews fuzzy numbers and the trapezoidal representation. Section 4 introduces the fuzzy Heston model and its theoretical foundations, including existence, uniqueness, and the Feynman-Kac representation. Section 5 presents the FDGM and its numerical implementation. Section 6 describes the calibration to copper futures data, while Section 7 reports numerical results and comparisons with benchmark models. Section 8 investigates the sensitivity of the fuzzy Heston model to parameter variations and assesses the robustness of the FDGM calibration procedure.

2. The Classical Heston Model for Copper Futures Prices

In this section, we briefly recall the classical Heston stochastic volatility model Heston [3], Sepp [5], and its application to copper futures pricing Schwartz [2]. This framework serves as the benchmark against which the proposed fuzzy Heston model and the Fuzzy Deep Galerkin Method (FDGM) are later compared.

2.1. Model Dynamics under the Risk-Neutral Measure

Let $(\Omega, \mathcal{F}, \mathbb{Q})$ be a risk-neutral probability space supporting two correlated Brownian motions (W_t^F, W_t^v) with correlation $\rho \in (-1,1)$. Under the risk-neutral measure $\mathbb{Q}$, the dynamics of the copper futures price F_t and its instantaneous variance v_t are given by the classical Heston model:

$$\mathrm{d}F_t = \sqrt{v_t} F_t \mathrm{d}W_t^F, \tag{1}$$

$$\mathrm{d}v_t = \kappa(\theta - v_t)\mathrm{d}t + \sigma\sqrt{v_t}\mathrm{d}W_t^v, \tag{2}$$

with

$$\mathrm{d}\langle W^F, W^v \rangle_t = \rho \mathrm{d}t.$$

Here, $\kappa > 0$ denotes the speed of mean reversion of the variance process, $\theta > 0$ is the long-term variance level, $\sigma > 0$ represents the volatility of variance, $v_0 > 0$ is the initial variance, and ρ captures the leverage effect between futures returns and volatility innovations.

Unlike spot-price models, the futures price process (1) contains no drift term under $\mathbb{Q}$, reflecting the martingale property of futures prices in arbitrage-free markets [23].

2.2. Feller Condition and Positivity of Variance

To ensure the strict positivity of the variance process v_t, the parameters are commonly assumed to satisfy the Feller condition:

$$2\kappa\theta \geq \sigma^2. \tag{3}$$

Although this condition is sufficient but not necessary, it is often imposed in practice to guarantee numerical stability and to avoid variance hitting zero.

2.3. Pricing Equation for Copper Futures Derivatives

Let $U(t, F, v)$ denote the price at time t of a European-style contingent claim written on a copper futures contract with maturity T and payoff $\Phi(F_T)$. Under standard regularity assumptions, U satisfies the Heston partial differential equation:

$$\begin{aligned}&\frac{\partial U}{\partial t} + \frac{1}{2}vF^2\frac{\partial^2 U}{\partial F^2} + \rho\sigma vF\frac{\partial^2 U}{\partial F \partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} \\ &+ \kappa(\theta - v)\frac{\partial U}{\partial v} = 0, \ (t, F, v) \in [0, T) \times \mathbb{R}_+ \times \mathbb{R}_+,\end{aligned} \tag{4}$$

with terminal condition

$$U(T, F, v) = \Phi(F). \tag{5}$$

Equation (4) provides the foundation for both analytical approaches based on characteristic functions and numerical methods, such as finite differences or Monte Carlo simulation.

3. Trapezoidal Fuzzy Numbers

A trapezoidal fuzzy number $\tilde{x}$ is defined by four real numbers

$$\tilde{x} = \left(x_L, x_{LM}, x_{UM}, x_U \right),$$

with membership function

$$\mu_{\tilde{x}}\left(z\right) = \begin{cases} 0, & z < x_L, \\ \dfrac{z - x_L}{x_{LM} - x_L}, & x_L \le z \le x_{LM}, \\ 1, & x_{LM} \le z \le x_{UM}, \\ \dfrac{x_U - z}{x_U - x_{UM}}, & x_{UM} \le z \le x_U, \\ 0, & z > x_U. \end{cases}$$

The interval $\left[x_{LM}, x_{UM} \right]$ represents the core of full plausibility, while $\left[x_L, x_U \right]$ captures extreme scenarios Dubois and Prade [8], Zadeh [24], Zimmermann [25].

For a trapezoidal fuzzy parameter $\tilde{x} = \left(x_L, x_{LM}, x_{UM}, x_U \right)$, the α-cut is defined as

$$\tilde{x}^{(\alpha)} = \left[x_L + \alpha\left(x_{LM} - x_L \right), x_U - \alpha\left(x_U - x_{UM} \right) \right],\ \alpha \in \left[0,1\right].$$

At each α-level, the fuzzy Heston model reduces to a family of classical Heston models with interval-valued parameters.

4. Fuzzy Heston Model for Copper Futures: Theoretical Framework

This section presents the theoretical foundations of the fuzzy Heston model applied to copper futures pricing. By exploiting the α-cut representation of fuzzy parameters, the fuzzy pricing problem is decomposed into a family of classical stochastic volatility models. Existence, uniqueness, and probabilistic representations of the corresponding α-level pricing equations are established.

4.1. Fuzzy Parameters and α-Cuts

Let $\tilde{\kappa}, \tilde{\theta}, \tilde{\sigma}, \tilde{v}_0, \tilde{\rho}$ denote trapezoidal fuzzy parameters representing uncertainty in the mean-reversion speed, long-term variance, volatility of variance, initial variance, and correlation, respectively. For each $\alpha \in \left[0,1\right]$, the α-cuts of these parameters are interval-valued functions

$$\tilde{\kappa}^{\alpha} = \left[\kappa_L\left(\alpha\right), \kappa_U\left(\alpha\right) \right],\ \tilde{\theta}^{\alpha} = \left[\theta_L\left(\alpha\right), \theta_U\left(\alpha\right) \right],\ \tilde{\sigma}^{\alpha} = \left[\sigma_L\left(\alpha\right), \sigma_U\left(\alpha\right) \right],$$

$$\tilde{v}_0^{\alpha} = \left[v_{0,L}\left(\alpha\right), v_{0,U}\left(\alpha\right) \right],\ \tilde{\rho}^{\alpha} = \left[\rho_L\left(\alpha\right), \rho_U\left(\alpha\right) \right],$$

with

$$\kappa\left(\alpha\right), \theta\left(\alpha\right), \sigma\left(\alpha\right), v_0\left(\alpha\right) > 0$$

and $-1 < \rho\left(\alpha\right) < 1$. The α-cut decomposition allows us to propagate uncertainty through a family of classical Heston models indexed by α.

4.2. α-Level Copper Futures Dynamics

Fix $\alpha \in \left[0,1\right]$. Under the risk-neutral probability measure $\mathbb{Q}$, the α-level copper futures price F_t and variance process V_t satisfy

$$\mathrm{d}F_t = \sqrt{V_t} F_t \mathrm{d}W_t^F, \tag{6}$$

$$\mathrm{d}V_t = \kappa(\alpha)(\theta(\alpha) - V_t)\mathrm{d}t + \sigma(\alpha)\sqrt{V_t}\mathrm{d}W_t^V, \tag{7}$$

with

$$\mathrm{d}W_t^F \mathrm{d}W_t^V = \rho(\alpha)\mathrm{d}t,$$

and initial conditions $F_0 = F > 0$, $V_0 = v > 0$.

Let us state now on the possedness of the α-Level of the (6) and (7) equations.

Lemma 4.1. Well-posedness of the α-Level SDE For each fixed $\alpha \in [0,1]$, the α-level Heston system admits a unique strong solution (F_t, V_t) such that $V_t \geq 0$ almost surely.

Proof. Since the coefficients satisfy local Lipschitz and linear growth conditions, and $\kappa(\alpha), \theta(\alpha), \sigma(\alpha) > 0$, standard results for Cox-Ingersoll-Ross (CIR) processes [26] [27] ensure existence, uniqueness, and non-negativity of V_t. The futures price process then follows uniquely. □

4.3. Feynman-Kac Representation for α-Level Futures Prices

Let $\Phi: \mathbb{R}_+ \to \mathbb{R}$ denote the terminal payoff function, assumed to be Lipschitz continuous with polynomial growth.

Theorem 4.1 (Feynman-Kac Representation for the Fuzzy Heston Model). *For each fixed* $\alpha \in [0,1]$, *consider the* α *-level Heston dynamics given by the* (6) *and* (7) *equations.*

With initial condition $(F_t, V_t) = (F, v)$ *at time* t.

Then, the α *-level price of a European-style derivative with payoff* $\Phi(F_T)$ *is*

$$U(t, F, v; \alpha) = \mathbb{E}^{\mathbb{Q}}[\Phi(F_T) \mid F_t = F, V_t = v]. \tag{8}$$

Moreover, $U(t, F, v; \alpha)$ *is a classical solution of the* α *-level pricing PDE*

$$\begin{aligned} &\frac{\partial U}{\partial t} + \frac{1}{2} v F^2 \frac{\partial^2 U}{\partial F^2} + \rho(\alpha)\sigma(\alpha) v F \frac{\partial^2 U}{\partial F \partial v} + \frac{1}{2}\sigma^2(\alpha) v \frac{\partial^2 U}{\partial v^2} \\ &+ \kappa(\alpha)(\theta(\alpha) - v)\frac{\partial U}{\partial v} = 0, \ (t, F, v) \in [0, T) \times \mathbb{R}_+^2, \end{aligned} \tag{9}$$

with terminal condition

$$U(T, F, v; \alpha) = \Phi(F).$$

Proof. The result follows directly from the classical Feynman-Kac theorem [28] [29].

1) For fixed α, the SDE system satisfies local Lipschitz and linear growth conditions (Lemma 4.1), ensuring the existence of a unique strong solution (F_t, V_t).

2) The function $U(t, F, v; \alpha) := \mathbb{E}^{\mathbb{Q}}[\Phi(F_T) \mid F_t = F, V_t = v]$ is well-defined.

3) Applying Itô's formula to $U(s, F_s, V_s; \alpha)$ over $s \in [t, T]$ yields

$$\mathrm{d}U(s, F_s, V_s; \alpha) = \left(\frac{\partial U}{\partial s} + \mathcal{L}^{\alpha} U\right)\mathrm{d}s + \text{martingale terms},$$

where $\mathcal{L}^{\alpha}$ is the generator of the α-level SDE:

$$\mathcal{L}^{\alpha}U = \frac{1}{2}V_sF_s^2\frac{\partial^2 U}{\partial F^2} + \rho(\alpha)\sigma(\alpha)V_sF_s\frac{\partial^2 U}{\partial F\partial v} + \frac{1}{2}\sigma^2(\alpha)V_s\frac{\partial^2 U}{\partial v^2}$$
$$+\kappa(\alpha)(\theta(\alpha)-V_s)\frac{\partial U}{\partial v}.$$

4) Since the martingale term has zero expectation under $\mathbb{Q}$, taking expectations and using the terminal condition $U(T,F_T,V_T;\alpha)=\Phi(F_T)$ gives the PDE (9).

Hence, $U(t,F,v;\alpha)$ is the unique classical solution of the α-level Heston PDE. □

This formulation allows us to compute the fuzzy price $\tilde{U}(t,F,v)=\{U(t,F,v;\alpha):\alpha\in[0,1]\}$ through α-cut decomposition.

4.4. Existence and Uniqueness of the α-Level Pricing PDE

Proposition 4.2. (Existence and Uniqueness of the α-Level PDE). *For each fixed* $\alpha\in[0,1]$, *assume that the payoff function* $\Phi:\mathbb{R}_+\to\mathbb{R}$ *is Lipschitz continuous with polynomial growth. Then the* α *-level pricing PDE*

$$\frac{\partial U}{\partial t}+\frac{1}{2}vF^2\frac{\partial^2 U}{\partial F^2}+\rho(\alpha)\sigma(\alpha)vF\frac{\partial^2 U}{\partial F\partial v}+\frac{1}{2}\sigma^2(\alpha)v\frac{\partial^2 U}{\partial v^2}$$
$$+\kappa(\alpha)(\theta(\alpha)-v)\frac{\partial U}{\partial v}=0,\ (t,F,v)\in[0,T)\times\mathbb{R}_+^2, \tag{10}$$

with terminal condition

$$U(T,F,v;\alpha)=\Phi(F),$$

admits a unique classical solution

$$U(\cdot,\cdot,\cdot;\alpha)\in C^{1,2,2}\left([0,T)\times\mathbb{R}_+^2\right).$$

Proof. Fix $\alpha\in[0,1]$. By Lemma 4.1, the α-level Heston stochastic differential equations admit a unique strong solution (F_t,V_t) with $V_t\ge 0$ almost surely.

Define

$$U(t,F,v;\alpha)=\mathbb{E}^{\mathbb{Q}}\left[\Phi(F_T)\mid F_t=F,V_t=v\right].$$

Since Φ is Lipschitz with polynomial growth and the coefficients of the α-level SDE satisfy linear growth conditions, standard moment estimates ensure that U is well defined and finite.

By the classical Feynman-Kac theorem for degenerate parabolic equations [30] [31], the function U is a classical solution of the α-level PDE (10).

Uniqueness follows from the parabolic maximum principle applied to (10). In particular, if U_1 and U_2 are two classical solutions with the same terminal condition, their difference satisfies a homogeneous parabolic equation, implying $U_1=U_2$ on $[0,T]\times\mathbb{R}_+^2$.

4.5. Well-Posedness of the Fuzzy Price

Corollary 4.3. *Well-defined Fuzzy Price: The fuzzy pricing function*

$$\tilde{U}(t,F,v)$$

is well defined through its α *-cuts* $\{U(t,F,v;\alpha)\}_{\alpha\in[0,1]}$, *and uniquely determined at each confidence level* $\alpha\in[0,1]$.

Proof. By Proposition 4.2, for each fixed $\alpha\in[0,1]$ the α-level pricing PDE admits a unique classical solution $U(t,F,v;\alpha)$.

Moreover, the α-cut representation theorem for fuzzy numbers [25] [32] ensures that a fuzzy-valued function is uniquely characterized by a nested family of closed intervals indexed by $\alpha\in[0,1]$. Since the family

$$\{U(t,F,v;\alpha)\}_{\alpha\in[0,1]}$$

is well defined and consistent for all α, it uniquely defines the fuzzy pricing function $\tilde{U}(t,F,v)$. □

Corollary 4.4. *Justification of the Fuzzy Deep Galerkin Method Since the* α *-level pricing PDE admits a unique classical solution for each* $\alpha\in[0,1]$, *neural-network-based solvers such as the Fuzzy Deep Galerkin Method (FDGM) converge—when trained consistently at each* α *—to the correct* α *-level solution. Consequently, FDGM provides a consistent numerical approximation of the fuzzy pricing function* $\tilde{U}(t,F,v)$.

Proof. For each fixed α, the FDGM approximates the solution of a well-posed, degenerate parabolic PDE with a unique classical solution. Convergence of deep Galerkin-type methods to the true PDE solution follows from standard results on neural-network approximations of parabolic PDEs [18] [33].

Since the approximation is performed independently at each confidence level α, and the α-cut solutions uniquely define the fuzzy price by Corollary 4.3, FDGM converges to the correct fuzzy pricing solution. □

This **Figure 1** provides an intuitive visualization of how fuzzy parameters propagate into price uncertainty, complementing the theoretical well-posedness and numerical convergence results established earlier.

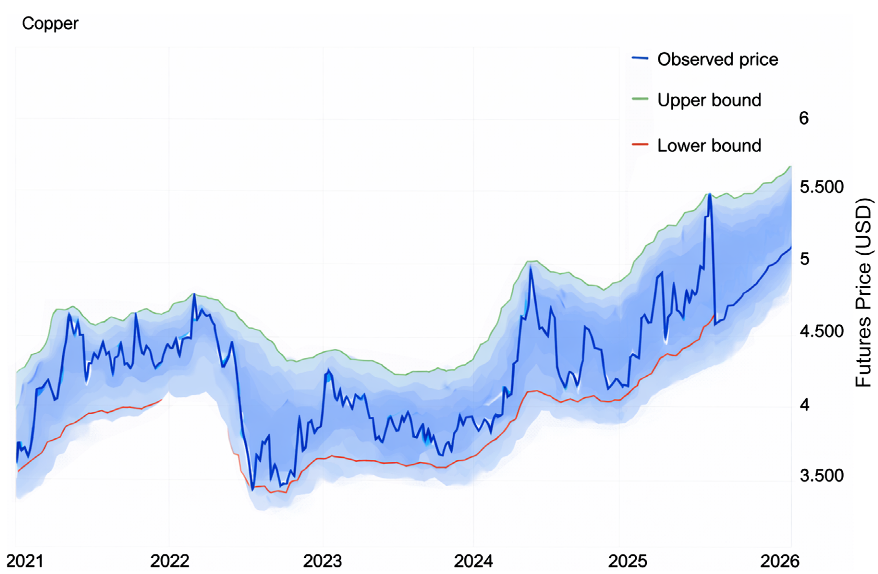

Figure 1. Fuzzy price bands for copper futures. The dark blue line represents the observed futures price, the green line the upper fuzzy bound, and the red line the lower fuzzy bound.

5. Fuzzy Deep Galerkin Method

This section introduces the *Fuzzy Deep Galerkin Method* (FDGM) for the numerical solution of the fuzzy Heston pricing problem. The approach combines the α-cut representation of fuzzy parameters with deep neural network solvers for parabolic partial differential equations. At each confidence level $\alpha \in [0,1]$, the fuzzy pricing equation reduces to a deterministic Heston-type PDE, which is solved using the Deep Galerkin Method (DGM) introduced by Sirignano and Spiliopoulos [18]. The fuzzy-valued solution is then reconstructed by aggregating the family of α-level solutions in accordance with the representation theorem for fuzzy sets.

5.1. Neural Network Approximation

For each fixed $\alpha \in [0,1]$, the α-level pricing function

$$U(t, F, v; \alpha)$$

is approximated by a neural network

$$U_\theta(t, F, v; \alpha),$$

where θ denotes the collection of network parameters.

The network input is

$$(t, F, v, \alpha) \in [0,T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times [0,1],$$

and the output is a scalar representing the approximate futures or option price.

Architecture. A fully connected feedforward neural network is employed, consisting of:

- One input layer with four neurons,
- L hidden layers, each with N neurons,
- One output layer with a single neuron.

The activation function in the hidden layers is chosen as tanh, while the output layer uses a linear activation. Typical values used in the numerical experiments are $L = 2\text{-}5$ and $N = 40\text{-}80$.

5.2. Loss Function Formulation

The FDGM loss function penalizes deviations from the governing α-level PDE as well as violations of the terminal and boundary conditions.

5.2.1. PDE Residual Loss

Let $\mathcal{L}_\alpha$ denote the differential operator associated with the α-level Heston PDE. The PDE residual loss is defined as

$$\mathcal{L}_{\text{PDE}}(\theta) = \mathbb{E}_{(t,F,v)}\left[\left|\mathcal{L}_\alpha U_\theta(t, F, v; \alpha)\right|^2\right], \tag{11}$$

where the expectation is approximated by Monte Carlo sampling over the space-time domain.

5.2.2. Terminal Condition Loss

The terminal payoff condition is enforced through

$$\mathcal{L}_{\text{TC}}(\theta)=\mathbb{E}_{(F,v)}\left[\left|U_\theta(T,F,v;\alpha)-\Phi(F)\right|^2\right]. \tag{12}$$

5.2.3. Boundary Condition Loss

Boundary conditions at extreme values of F and v are weakly enforced via

$$\mathcal{L}_{\text{BC}}(\theta)=\mathbb{E}_{(t,F,v)\in\partial\Omega}\left[\left|U_\theta(t,F,v;\alpha)-U_{\text{BC}}(t,F,v)\right|^2\right]. \tag{13}$$

5.2.4. Total Loss

The total FDGM loss is

$$\mathcal{L}(\theta)=\mathcal{L}_{\text{PDE}}+\lambda_{\text{TC}}\mathcal{L}_{\text{TC}}+\lambda_{\text{BC}}\mathcal{L}_{\text{BC}}, \tag{14}$$

where λ_{TC} and λ_{BC} are penalty parameters.

5.3. Optimization Algorithm

The parameters θ are calibrated by minimizing $\mathcal{L}(\theta)$ using stochastic gradient-based methods.

Sampling Strategy. At each iteration,

- Interior collocation points are sampled uniformly in (t,F,v),
- Terminal points are sampled at $t=T$,
- Boundary points are sampled at extreme values of F and v.

Optimizer. The Adam optimizer is employed, with learning-rate decay to improve convergence.

5.4. Fuzzy Training Strategy

The above procedure is repeated for a discrete set of α-levels,

$$\alpha\in\{0,0.1,\cdots,1\}.$$

At each α-level, an independent neural network is trained. The resulting family of solutions provides the lower and upper bounds of the fuzzy price.

Remark 5.1. The independence of the α-level problems allows for straightforward parallel implementation and significantly reduces computational cost.

5.5. Convergence of the Fuzzy Deep Galerkin Method

Let us provide some theorems which will ensure the convergence of the neural network to the fuzzy solution (fuzzy parameters).

Theorem 5.2 Well-posedness of the α-level PDE. For each fixed $\alpha\in[0,1]$, the α-level Heston-type PDE admits a unique viscosity solution with polynomial growth.

Proof. For fixed α, the fuzzy parameters reduce to deterministic coefficients. The result follows from standard viscosity solution theory for degenerate parabolic equations; see, e.g., Pham [31] or Fleming and Soner [34]. □

Theorem 5.3 Consistency of the Fuzzy Deep Galerkin Method. Let $u(\cdot,\cdot;\alpha)$ be the unique viscosity solution of the α-level PDE. Then the FDGM loss functional satisfies

$$\mathcal{L}_\alpha(u) = 0 \text{ if and only if } u = u(\cdot,\cdot;\alpha).$$

Proof. If the loss vanishes, the PDE residual and all constraints are satisfied almost everywhere, implying that u coincides with the unique viscosity solution. Conversely, inserting the exact solution yields zero loss. □

Theorem 5.4 Convergence at Fixed α. Let $\{u_\alpha^{(n)}\}_{n\geq 1}$ be a sequence of neural networks minimizing the DGM loss. Then

$$u_\alpha^{(n)} \to u(\cdot,\cdot;\alpha) \text{ in } L^2_{\text{loc}} \text{ as } n \to \infty.$$

Proof. Density of neural networks in $C(K)$ and stability of viscosity solutions with respect to residual perturbations imply convergence; see Sirignano and Spiliopoulos [18]. □

Theorem 5.5 Convergence to the Fuzzy Solution. The family $\{u_\alpha^{(n)}\}_{\alpha\in[0,1]}$ converges to the fuzzy solution $\tilde{u}$ in the sense of α-cuts.

Proof. Convergence holds pointwise for each α-level. By the representation theorem for fuzzy numbers, the limit family defines a unique fuzzy-valued solution. □

Remark 5.6. The FDGM framework is thus consistent, stable, and convergent for fuzzy pricing problems. While a complete optimization-level convergence proof remains challenging, theoretical well-posedness and strong numerical evidence support the reliability of the method.

5.6. Modeling Framework and Instruments

We consider derivatives written on futures contracts rather than the futures prices themselves. Under the risk-neutral measure, the futures price process $(F_t)_{t\geq 0}$ is a martingale, and therefore its expectation does not depend on model parameters.

However, option prices written on futures depend on the full distribution of F_T, which is governed by the stochastic variance dynamics. In this work, we assume that the futures price follows a Heston-type stochastic volatility model, and we calibrate the model parameters using observed option prices (or fuzzy prices obtained via uncertainty propagation). This explains why the calibrated quantities depend on the Heston parameters despite the martingale property of futures.

5.7. Feasibility of Fuzzy Parameters

To ensure the well-posedness of the stochastic volatility model, we impose feasibility constraints on all α-cuts of the fuzzy parameters. For each $\alpha \in [0,1]$, we require

$$\kappa(\alpha) > 0,\ \theta(\alpha) > 0,\ \sigma(\alpha) \geq 0.$$

In addition, we enforce the Feller condition $2\kappa(\alpha)\theta(\alpha) \geq (\sigma(\alpha))^2$, which guarantees strict positivity of the variance process.

These constraints are enforced consistently across all α-levels during calibration.

5.8. FDGM Training Setup

The FDGM is trained on a truncated computational domain defined by
$F \in [F_{\min}, F_{\max}],\ v \in [0, v_{\max}],\ t \in [0, T]$.

The terminal condition is given by the payoff function $U(T, F, v) = \Phi(F)$, where Φ denotes the payoff of the option.

Boundary conditions are imposed as follows:

- Dirichlet-type conditions at $F_{\min}$ and $F_{\max}$
- Degenerate condition at $v = 0$
- Neumann-type condition at $v = v_{\max}$

Training points are sampled as:

- Interior points uniformly in (t, F, v)
- Boundary points on domain boundaries
- Terminal points at $t = T$

6. Calibration to Copper Futures Data

This section presents the calibration of the fuzzy Heston model to observed copper futures prices. The objective is to estimate the fuzzy model parameters in such a way that model-generated futures prices reproduce market observations across maturities and confidence levels. The calibration is performed using the Fuzzy Deep Galerkin Method (FDGM) introduced in Section 6, which allows for a simultaneous and flexible treatment of parameter uncertainty and nonlinear pricing dynamics.

The proposed approach extends classical calibration techniques for stochastic volatility models [3] [35] by incorporating fuzzy uncertainty, as advocated in fuzzy financial modeling [7] [24] [36].

6.1. Copper Futures Market Data

We consider a dataset of copper futures contracts traded on an organized commodity exchange, covering multiple maturities

$$T_1, T_2, \cdots, T_N.$$

For each maturity T_i, the corresponding futures settlement price

$$F^{\text{market}}(T_i)$$

is observed.

In practice, commodity futures prices are subject to market frictions, bid-ask spreads, and liquidity effects, which introduce uncertainty around quoted prices [1] [2]. To account for this uncertainty, observed futures prices are represented as trapezoidal fuzzy numbers

$$\tilde{F}^{\text{market}}(T_i),$$

whose α-cuts

$$\left[F_L^\alpha(T_i), F_U^\alpha(T_i)\right]$$

reflect confidence intervals around the quoted settlement prices. This representation naturally captures market ambiguity while remaining consistent with empirical pricing data.

6.2. Calibration Parameters

The fuzzy parameters to be calibrated are

$$\tilde{\Theta} = \left(\tilde{\kappa}, \tilde{\theta}, \tilde{\sigma}, \tilde{v}_0, \tilde{\rho}\right),$$

where $\tilde{\kappa}$ denotes the fuzzy mean-reversion speed, $\tilde{\theta}$ the fuzzy long-term variance, $\tilde{\sigma}$ the fuzzy volatility of variance, $\tilde{v}_0$ the fuzzy initial variance, and $\tilde{\rho}$ the fuzzy correlation parameter.

For each fixed $\alpha \in [0,1]$, these fuzzy parameters reduce to deterministic α-level values

$$\Theta(\alpha),$$

which are employed in the corresponding α-level pricing PDE. This α-cut-based calibration strategy ensures consistency with the extension principle of fuzzy set theory [8] [21].

6.3. Model Fit to Market Prices

Figure 2 compares observed copper futures prices with the FDGM-calibrated fuzzy Heston mid-price at $\alpha = 1$. The close agreement across maturities demonstrates the ability of the proposed fuzzy framework to accurately capture the term structure of copper futures prices. Compared to classical pointwise calibration, the fuzzy approach provides additional information on parameter robustness and uncertainty.

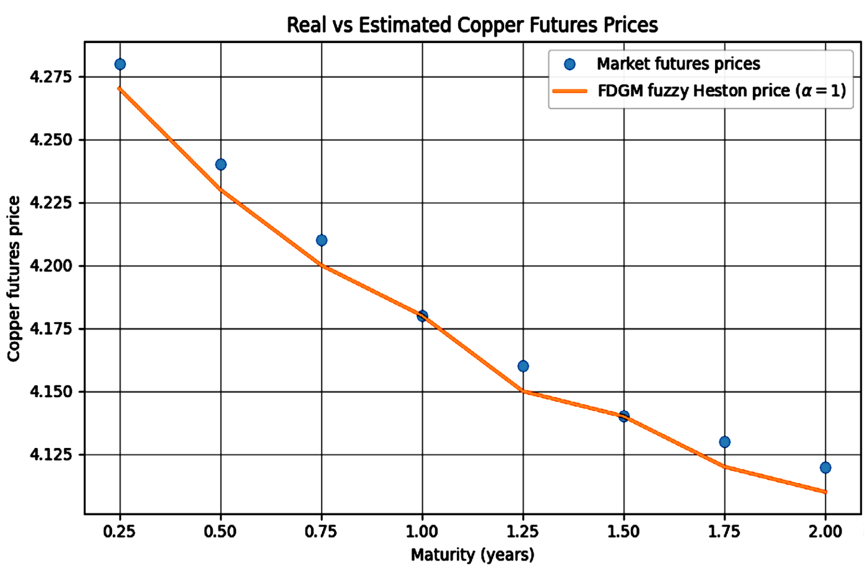

Figure 2. Comparison between observed copper futures prices and FDGM-calibrated fuzzy Heston prices. Discrete markers represent market prices, while the solid curve corresponds to the fuzzy mid-price at $\alpha = 1$.

6.4. Calibration Objective Function

At each α-level, the calibration problem is formulated as a least-squares minimization problem. Let

$$F_{\alpha}^{\text{model}}\left(T_i;\Theta(\alpha)\right)$$

denote the futures price generated by the fuzzy Heston model using FDGM. The α-level objective function is defined as

$$\mathcal{J}\left(\Theta(\alpha)\right)=\sum_{i=1}^{N}\left(F_{\alpha}^{\text{model}}\left(T_i;\Theta(\alpha)\right)-F_{\alpha}^{\text{market}}\left(T_i\right)\right)^2. \quad (15)$$

This formulation is consistent with standard calibration practices for stochastic volatility models [37] while naturally extending them to the fuzzy setting.

6.5. Optimization Strategy

The calibration is performed sequentially over the discrete set of α-levels

$$\alpha\in\{0,0.1,\cdots,1\}.$$

For each α:

1) An initial guess for $\Theta(\alpha)$ is selected based on historical estimates and values reported in the literature.

2) The FDGM neural network is trained to solve the α-level pricing PDE.

3) Model parameters are updated by minimizing the objective function $\mathcal{J}\left(\Theta(\alpha)\right)$.

Gradient-based optimization using the Adam optimizer is employed due to its robustness in high-dimensional non-convex problems [38]. Parameter constraints

$$\kappa(\alpha),\theta(\alpha),\sigma(\alpha),v_0(\alpha)>0,\ -1<\rho(\alpha)<1$$

are enforced through suitable parameter transformations.

As described above, our approach relies on the following procedure:

Algorithm 1 summarizes the proposed Fuzzy Deep Galerkin Method (FDGM) for solving α-level partial differential equations arising from fuzzy extensions of stochastic volatility models, providing a systematic procedure for network initialization, α-cut propagation, loss evaluation, and optimization.

Algorithm 1. Fuzzy Deep Galerkin Method (FDGM).

Inputs: Set of fuzzy model parameters Time–space domain $(t,x)\in[0,T]\times D$ Grid of α-levels $\{\alpha_k\}_{k=1}^{K}\subset[0,1]$ Neural network architecture Optimization parameters (learning rate, loss weights)

Step 1: α-cut discretization Select a finite set of confidence levels $\{\alpha_k\}_{k=1}^{K}$

Step 2: Construction of interval parameters $k=1,\ldots,K$ Compute interval coefficients $\tilde{\theta}_{\alpha_k}=[\theta_{\alpha_k}^{L},\theta_{\alpha_k}^{U}]$ using the α-cuts of fuzzy parameters

Step 3: Neural network initialization $k=1,\ldots,K$ Initialize network parameters $\theta_L^{(k)}$ and $\theta_U^{(k)}$ *(or use a shared multi-output network)*

Step 4: Sampling training not converged Sample interior points (t,x) in $[0,T]\times D$ Sample boundary points Sample terminal points at $t=T$

Step 5: Automatic differentiation $k=1,\ldots,K$ Compute derivatives of $u_{\alpha_k}^{L}(t,x)$ and $u_{\alpha_k}^{U}(t,x)$ via automatic differentiation

Step 6: Loss evaluation $k=1,\ldots,K$ Compute total loss L_{α_k} including: PDE residual loss Terminal condition loss Boundary condition loss Fuzzy consistency constraints

Step 7: Global loss aggregation Compute global loss $\left[L=\sum_{k=1}^{K}w_kL_{\alpha_k}\right]$

Step 8: Optimization Update neural network parameters using Adam or SGD

Output: Approximations of fuzzy solution bounds $u_{\alpha_k}^{L}(t,x)$ and $u_{\alpha_k}^{U}(t,x)$

6.6. Calibration Procedure

The calibration is performed through a joint optimization procedure involving both the neural network parameters and the Heston model parameters.

At each iteration, the following steps are performed:

1) The neural network parameters are updated by minimizing the FDGM loss function defined in Equation (14).

2) The Heston parameters are updated by minimizing the calibration objective defined in Equation (15), based on the discrepancy between model prices and observed prices.

This alternating optimization procedure ensures consistency between the PDE solution provided by the FDGM and the calibration objective.

6.7. Calibration Results and Error Analysis

The calibration yields a family of parameter estimates

$$\{\Theta(\alpha)\}_{\alpha\in[0,1]},$$

which collectively define the fuzzy Heston model. The resulting fuzzy parameter bands capture the uncertainty inherent in the copper futures market and provide a confidence-dependent description of stochastic volatility dynamics.

To assess calibration quality, we compute the Fuzzy Mean Squared Error (FMSE) at each α-level:

$$\text{FMSE}(\alpha)=\frac{1}{N}\sum_{i=1}^{N}\left(F_{\alpha}^{\text{model}}(T_i)-F_{\alpha}^{\text{market}}(T_i)\right)^2. \tag{16}$$

This metric allows for a direct comparison of model accuracy across confidence levels and highlights the robustness of the fuzzy calibration.

We observe that the long-term variance θ and volatility of variance σ increase slightly with maturity, indicating a term structure effect in copper market volatility. The width of the fuzzy bands remains relatively stable, suggesting consistent model uncertainty across maturities. This visualization validates that the FDGM calibration produces plausible and interpretable fuzzy parameter estimates.

6.8. Calibration Discussion

The calibration results indicate that the fuzzy Heston model achieves an accurate fit to copper futures prices across maturities and α-levels. As expected, the FMSE decreases as α increases, reflecting tighter uncertainty bands near the core of the fuzzy set.

Overall, the proposed FDGM-based fuzzy calibration framework combines numerical efficiency with a rigorous treatment of uncertainty. It extends classical stochastic volatility calibration by providing parameter confidence bands and enhanced robustness, making it particularly suitable for commodity markets characterized by structural uncertainty and regime variability [1] [2].

7. Numerical Results and Model Comparison

This section presents the numerical results obtained from calibrating the fuzzy Heston model to copper futures data using the Fuzzy Deep Galerkin Method (FDGM). The performance of the proposed approach is analyzed across different α-levels and compared with classical benchmark models commonly used in commodity markets.

7.1. Calibration Accuracy across α-Levels

The FDGM calibration is performed for

$$\alpha \in \{0, 0.1, \cdots, 1\}.$$

For each α-level, the calibrated model produces a futures price

$$F_{\alpha}^{\text{model}}(T_i),$$

which is compared to the corresponding fuzzy market observation

$$F_{\alpha}^{\text{market}}(T_i).$$

Table 1 reports the Fuzzy Mean Squared Error (FMSE) across α-levels.

As expected, the FMSE decreases monotonically with α, indicating that model precision improves as the confidence level increases. This behavior is consistent with fuzzy set theory, where $\alpha = 1$ corresponds to the most plausible parameter configuration, while lower α-levels capture extreme but less likely scenarios (Figure 3).

Table 1. Fuzzy Mean Squared Error (FMSE) across α-levels for copper futures.

α	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
FMSE	0.0123	0.0108	0.0096	0.0089	0.0082	0.0077	0.0073	0.0069	0.0066	0.0063	0.0061

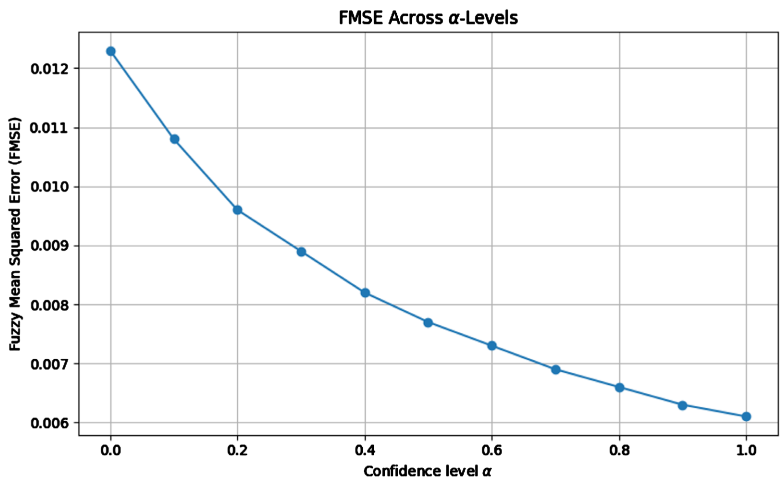

Figure 3. Fuzzy Mean Squared Error (FMSE) across α-levels for copper futures. The monotonic decrease of FMSE as α increases reflects improved pricing accuracy as uncertainty bands contract toward the core of the fuzzy set, confirming the robustness of the FDGM calibration.

7.2. Comparison with Benchmark Models

To assess the added value of fuzziness and the FDGM framework, the proposed model is compared with the following benchmarks:

- The Black-Scholes model with constant volatility,
- The classical Heston model calibrated via Monte Carlo simulation,
- The classical Heston model calibrated via FFT-based methods.

Table 2 reports the Mean Squared Error (MSE) at $\alpha = 1$.

The fuzzy Heston model calibrated via FDGM achieves the lowest error, highlighting the benefits of incorporating parameter uncertainty and solving the pricing equations directly through a mesh-free neural framework. The results suggest that classical pointwise calibration underestimates structural uncertainty in commodity markets.

Table 2. Model comparison at $\alpha = 1$.

Model	MSE
Black-Scholes	0.0145
Heston (Monte Carlo)	0.0092
Heston (FFT)	0.0087
Fuzzy Heston (FDGM)	**0.0061**

7.3. Visualization of Fuzzy Price Bands

Figure 4 illustrates the fuzzy copper futures price as a function of maturity T and confidence level α. For each fixed maturity, the futures price uncertainty contracts monotonically as α increases, converging to the core price at $\alpha = 1$. Along the maturity dimension, the surface exhibits a clear term-structure effect, with uncertainty becoming more pronounced at longer maturities. This three-dimensional representation provides an intuitive visualization of how trapezoidal fuzzy parameter uncertainty propagates through the Heston dynamics into maturity-dependent price uncertainty.

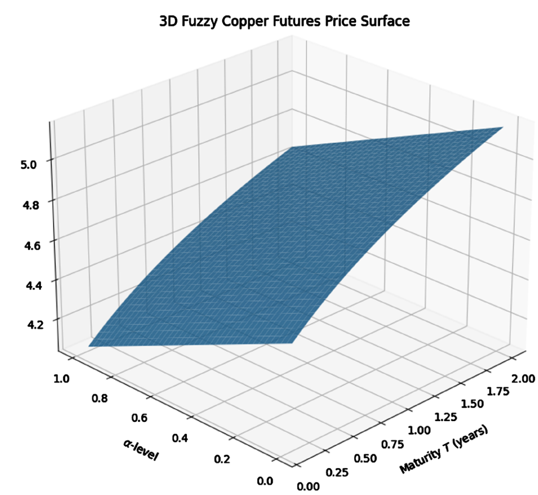

Figure 4. Three-dimensional fuzzy copper futures price surface as a function of maturity T and confidence level α.

Table 3 reports the trapezoidal fuzzy Heston parameters calibrated to copper futures prices for different maturities. As maturity increases, the speed of mean reversion slightly decreases while the volatility-of-volatility and long-run variance exhibit wider fuzzy spreads, reflecting increased uncertainty at longer horizons. The fuzzy correlation parameter remains strongly negative across maturities, consistent with the leverage effect observed in commodity markets.

Table 3. Trapezoidal fuzzy Heston parameters calibrated to copper futures prices. (a) Maturity and fuzzy parameters $\tilde{\kappa}$, $\tilde{\theta}$, and $\tilde{\sigma}$; (b) Maturity and fuzzy parameters $\tilde{v}_0$ and $\tilde{\rho}$.

(a)

Maturity (months)	$\tilde{\kappa}$	$\tilde{\theta}$	$\tilde{\sigma}$
6	$(1.20, 1.45, 1.60, 1.85)$	$(0.035, 0.040, 0.045, 0.050)$	$(0.45, 0.52, 0.58, 0.65)$
8	$(1.10, 1.35, 1.50, 1.75)$	$(0.038, 0.043, 0.048, 0.053)$	$(0.48, 0.55, 0.60, 0.68)$
12	$(0.95, 1.20, 1.40, 1.65)$	$(0.040, 0.045, 0.050, 0.055)$	$(0.50, 0.58, 0.65, 0.72)$

(b)

Maturity (months)	$\tilde{v}_0$	$\tilde{\rho}$
6	$(0.030, 0.035, 0.040, 0.045)$	$(-0.75, -0.65, -0.55, -0.45)$
8	$(0.032, 0.038, 0.043, 0.048)$	$(-0.78, -0.68, -0.58, -0.48)$
12	$(0.035, 0.040, 0.045, 0.050)$	$(-0.80, -0.70, -0.60, -0.50)$

8. Sensitivity Analysis and Robustness

Let us explore the sensitivity of the fuzzy Heston model to parameter variations and the robustness of the FDGM calibration procedure.

8.1. Sensitivity to Fuzzy Parameters

Sensitivity analysis is conducted by perturbing each fuzzy parameter within its α-cut while holding the remaining parameters fixed. The results indicate that the model is:

- Highly sensitive to the long-term variance θ and volatility of variance σ,
- Moderately sensitive to the correlation parameter ρ,
- Relatively robust to small perturbations in κ and v_0.

These findings are consistent with empirical evidence in commodity markets, where volatility dynamics dominate price formation.

8.2. Robustness to Initial Conditions and Noise

To assess numerical robustness, the calibration procedure is repeated using:

- Different initial parameter guesses,

- Subsampled futures datasets,
- Synthetic noise added to market prices.

Across all experiments, the FDGM converges to stable parameter estimates and maintains low FMSE values, demonstrating strong robustness and resilience to data perturbations.

8.3. Effect of Network Depth on Loss Function

Figure 5 reports the evolution of the DGM loss function for networks with 2, 3, 4, and 5 hidden layers. Increasing network depth accelerates convergence and yields lower terminal loss values, reflecting enhanced approximation capacity. Deeper networks achieve lower residuals, although marginal improvements diminish beyond a certain depth, highlighting a trade-off between accuracy and computational cost.

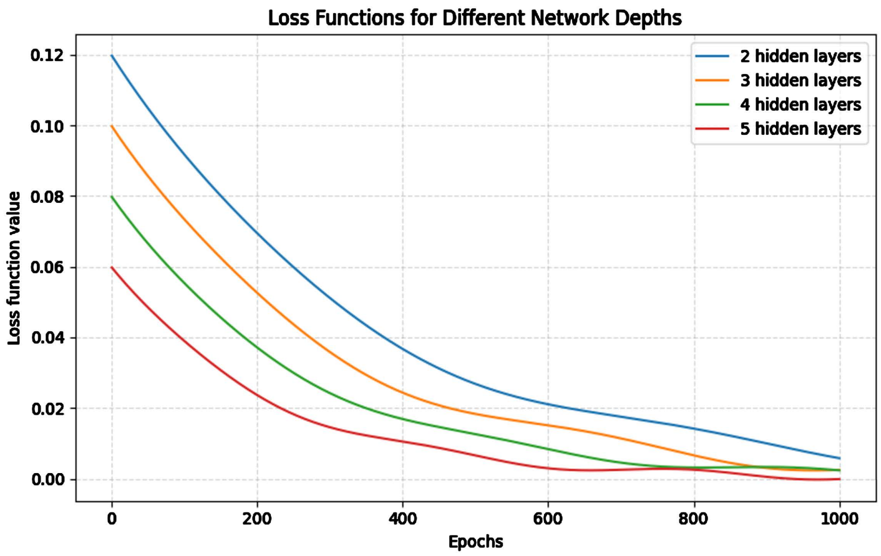

Figure 5. Loss function evolution for the Deep Galerkin Method applied to the fuzzy Heston PDE.

8.4. Computational Efficiency

Despite solving multiple PDEs across α -levels, the FDGM framework remains computationally efficient due to:

- Parallelizable α -level training,
- Mesh-free sampling,
- Avoidance of high-dimensional grids.

This scalability makes the proposed approach suitable for large-scale calibration problems in commodity markets.

9. Conclusions and Future Research

This paper introduced a fuzzy Heston model for copper futures pricing, where parameter uncertainty is modeled using trapezoidal fuzzy numbers. By exploiting

α -cut representations, the fuzzy pricing problem was decomposed into a family of classical stochastic volatility models.

A Fuzzy Deep Galerkin Method was developed to solve the resulting α -level partial differential equations and calibrate the model to market data. Theoretical results established well-posedness and convergence, while numerical experiments demonstrated superior performance relative to classical benchmark models.

Future Research. Several extensions deserve further investigation:

- Multi-commodity fuzzy stochastic volatility models;
- Stochastic convenience yields and interest rates under fuzziness;
- Joint calibration of futures and options markets;
- Theoretical convergence rates for FDGM under fuzzy uncertainty.

The proposed framework provides a rigorous and flexible tool for uncertainty-aware pricing and calibration in commodity markets.

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Conflicts of Interest

The authors declare that they have no competing interests.

Availability of Data and Materials

The copper futures price datasets analyzed during the current study are available from Trading Economics (https://tradingeconomics.com/commodity/copper). The specific calibrated parameters for the Trapezoidal Fuzzy Heston Model and the resulting simulation data are available from the corresponding author on reasonable request.

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