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Lazare Carnot's Mechanics: The Beginning of Group Theory in Science

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Abstract

Present paper quickly reviews the main characteristic features of Lazare Carnot's formulation of mechanics according to the results of the research of the last four decades. Three basic features of its theoretical development are illustrated: 1) Its starting point as declared by Carnot himself, i.e. the principle of virtual works modified in a suitable version for the case of the impact of bodies; the theoretical treatment of this phenomenon avoids the use of the main basic notion of Newton's theory of mechanics, i.e. the force-cause; 2) The recognition of two basic choices on two dichotomies concerning the entire theoretical physics; his choices are the alternatives ones to Newton's; 3) The introduction of the group theory in theoretical physics; Carnot was proud to have given birth to "an intermediate science between geometry and mechanics". The latter novelty is compared with *i*) René-Just Haüy's introduction (occurred one year later) of symmetries into crystallography; *ii*) Evariste Galois' introduction (occurred fifty years later) of group theory into mathematics; *iii*) Pierre Curie's introduction (occurred a century later) of symmetry breaking into theoretical physics; *iv*) Albert Einstein's 1905 paper introducing Lorenz group into mechanics. The open problems of a rational re-construction of L. Carnot's formulation according to two alternative basic choices are suggested.

Keywords

Lazare Carnot, Group Theory

1. The Historiography about Lazare Carnot's Scientific Works

The historical studies about the scientist Lazare Carnot grew slowly in time although he was one of the most important scientist of his time since he had written celebrated books on all scientific theories of his time, i.e. geometry, calculus and mechanics. Maybe the studies on him were difficult because he intermingled, as no other scientist, three skills: science, politics and military strategy; in particular, he was a leading actor of French revolution and also of the post-revolution time.

First of all, his biography. In the '50's Michel Reinhard's two volumes offered a complete view of his life (Reinhard, 1950, 1952).

Second, his scientific works. His writing for responding to a call for papers by the Academy of Sciences of Berlin on the subject of the infinity was discovered in the '70s by Adolf P. Youschkevitch (Youschkevitch, 1971: pp. 149-168). Some minor writings have been discovered in the 90's (Carnot, 1990). Hence, two centuries passed before completing the list of his wide production on science was completed.

Third, the contents of his scientific works. Present studies on the history of science of the years around 1800 are largely incomplete. No surprise if also the studies on Carnot's scientific works are few. At his time the works on calculus were famous (just after the French edition his last book (Carnot, 1813) was translated in England and also in Russia[1]). They received some attention by the historians because Carnot as first suggested an anti-metaphysical formulation of calculus, actually, an alternative conception of infinitesimal analysis. Charles Boyer's book offered a clever appraisal of both his calculus and his geometry (Boyer, 1991: pp. 430-450). But the many works on geometry still wait for a panoramic view[2].

Before the 70's, Lazare's formulation of work in mechanics has been considered as the beginning of a kind of theoretical physics, which later was called "technical physics", i.e. an engineers' field of studies receiving its principles from the usual theoretical physics. However, his theoretical innovations are so many that they do not allow to consider it as a theory of only machines. This point was emphasized by Charles Coulson Gillispie (Gillispie, 1971: chap. 4) by means of an accurate review of his first two works on mechanics. Gillispie also deserves the merits of having published his first two papers on mechanics (Carnot, 1971) and moreover having stressed that the thermodynamics founded by Lazare's son, Sadi, originated from his formulation of mechanics. In sum, the subject of L. Carnot's mechanics was extensively studied. A recent book (Gillispie & Pisano, 2013) reiterated and widened the subject of the Carnot's scientific works, included that on geometry and calculus (chap. V)[3].

[1]It is highly likelihood that N.I. Lobachevsky, the first inventor of non-Euclidean geometry, advocated the translation of this book into Russian language; as a fact it was published in the little and far town, Kazan, where Lobachevsky lived. He well knew L. Carnot's geometry. He quoted Carnot notwithstanding the censorship of his times erased the names of the revolutionary men (Bazhanov & Drago, 1998).

[2]In particular, the complex treatise, *Géométrie de Position* (Carnot, 1803b), is little known. It starts from a dissertation on the two ancient methods, analysis and synthesis; it suggests a new kind of organization of the geometry as based on a problem, instead of axioms; its basic problem was the validity of the negative numbers within geometry or not; in order to solve the basic problem and ~~the~~ related problems he introduces a cumbersome computational technique (Boiano & Drago, 2002).

[3]A detailed review of all the connections between Lazare's and Sadi's scientific work is given by the paper (Drago & Iacono, 2009). The decisive role played by Lazare's mechanics in suggesting a general theoretical framework to the thermodynamic theory of his son is treated by (Drago, 2026). A different analysis is that of (Schubring, 2005: chap. V); his aim, however, is to highlight the subjective notions of the title of his book, rather than the foundational and structural aspects of science.

Fourth, the appraisals on Lazare's scientific works. An accurate and definitive appraisal of them is an open question[4]. In particular, those on mechanics have to be evaluated first of all in comparison Newton's mechanics: is Carnot's theory an appropriate formulation to technical applications? Or is it a lateral theory with respect to Newton's theory? Or is it an alternative theory to Newton's?

In the following sections 2-5 I will list the characteristic features supporting the latter case: in particular, I will recognize the fundamental choices of this theory which are alternative to that of Newtonian paradigm (section 6); in sections 7-8 I will illustrate Carnot's mathematical technique of symmetries. At last (section 9) a suggestion for re-constructing in a rational way this formulation of mechanics is offered.

2. The Many Innovations of Carnot's Mechanics

The most relevant studies on Carnot's theory of mechanics are the following ones.

The book (Scott, 1970) made apparent the following features of this formulation: the relevance of the impact of bodies for the historical development of TP; the conservation of energy which subsequently has been long time neglected; the great relevance for a theory of mechanical machines; the emergences of the following ideas: system's state, reversible transformations and cycle of operations.

Gillispie (1971: chap. IV) reviewed in a detailed way the contents of all Carnot's writings on mechanics.

Some innovative studies followed. The papers (Drago & Manno, 1986; Drago & Manno, 1989; Drago, 1990a; Drago, 2004) illustrated some more theoretical novelties of L. Carnot's formulation of mechanics. The preface to the Italian translation of *Essai* (Drago & Manno, 1994: "Prefazione", pp. ix-xx) summarized and elaborated the contents of these papers.

Owing to the bounds of space, I do not list all the several innovations of Carnot's mechanics illustrated by the above writings also because several of them are presented by two specific chapters of (Gillispie & Pisano, 2013: chaps. 2-4). Rather, I underline the novelties of great theoretical relevance which do not appear in this book, however illustrated by the subsequent book (Drago, 2017).

1) Also Carnot's first book on mechanics considers machines in the most general terms as possible, that is, according to the following definition: a "machine" is everything that transmits movement by impact or by contact mediated by wires or rods. Since according to him space is full and he rejected the action at distance of the gravitational force as an unexplained notion, then no mechanical phenomenon is outside such a theory of machines. It is therefore defined as "the science of the communication of motion" between intermediary bodies (Carnot, 1783:

[4]The historians' researches on him are disturbed by his fame of "Régicide" (because his vote was decisive for deciding to execute king Louis XVI). Therefore in the scientific community (and more in general in the reactionary world) he was the "unnammed". Evidence for this obscuring Lazare Carnot was collected by Grattan-Guinness (1990: p. 1971) and me. It is enough to recall that Cauchy (1830-31) wrote that were wrong the theses of "l'auteur des réflexions sur la métaphysique de l'analyse infinitésimal", without any more qualification of this author, clearly Lazare Carnot who had written a celebrated book with this title.

sections viii and ix; Gillispie & Pisano, 2013: p. 119).

2) By following Leibniz's attempt of building an alternative formulation of mechanics to Newton's, in the foundations of Carnot's mechanics calculus is missing, because, as Leibniz put it, "the empirical facts have to be explained by the empirical facts" (Leibniz, 1677). By following again Leibniz's attempt of new mechanics, he rejects the notions of absolute space and time, and rather considers the relativity of motion and the relative reference systems; he states the inertia principle as the indifference of a body to rest or uniform motion, establishes the laws of elastic impact of bodies, the impossibility of a perpetual motion, the Torricelli's principle, and conservation of energy in general for all kinds of bodies[5], i.e. the principle that originated the first principle of thermodynamics and at present is basic for the entire theoretical physics.

3) He rejects the "metaphysical and obscure notion of force-cause" (Carnot, 1803a: p. xii) for rather considering mainly weights; he restores the mass of all elements of a machines at (Carnot, 1783: section viii) that instead at that time were dismissed for theoretical purposes.

4) Before Lazare the metaphysical notion of a force-cause was dominant; it denied the possibility of a force-cause acting against the motion of a body; hence, friction and other forces composing with velocity an obtuse angle were ignored. Instead, L. Carnot introduced in a clear way the distinction between the active forces and the resistant forces, according to the angle they have with the velocity of the body at issue (Carnot, 1783: preface).

5) By first suggesting an index of elasticity L. Carnot generalized the laws for the impact of elastic bodies to all kinds of bodies, i.e. both elastic bodies and soft bodies (Carnot, 1783: section xxvi). This index is the same of the modern one, apart its numerical values.

6) He suggested a new version of inertia principle, i.e. the physical principle differentiating modern science from ancient science and also representing "the philosopher door to philosophy of physics" (Hanson, 1965: title). His version is entirely experimental (Dugas, 1950: pp. 309-311) and does not present tautologies (which instead characterize Newton's version once it is purged of the notions of absolute space and time), i.e. the tautologies among the definitions of reference system, inertial motion and uniform motion.

7) Although he preferred a theoretical organization aimed at solving a problem (as the 1783 book is organized) in 1803 book he deliberately re-formulated his theory as deriving it from some principles, whoever called by him "hypotheses"[6]. They mainly concern the impact of bodies[7]. A detailed analysis and a comparison

[5]Leibniz concluded a reflection on conservation of energy by extending it to the various kinds of bodies and even the environment (Leibniz, 1693: section 27).

[6]Probably he had to submit the second edition of the book to the judgment of those scientists who, like Laplace, considered the deductive organization and Newton's principles to be the pinnacle of theoretical physics. Therefore, in this work Carnot decided to accept the challenge of also organizing mechanics by deducing it from principles (and so to start the theory from static): but he did so in an original way with respect to the development of Newton's theory from its principles.

[7]He added a section to lucidly explain an application which at his time was not clear: this principle holds true also for "animated bodies", i.e. animals (Carnot, 1803a: p. 246).

of these principles with Newton's ones is offered by (Drago & Manno, 1989)[8]. In particular, Carnot makes use of doubly negated propositions not equivalent to the corresponding affirmative propositions for the lack of evidence of the latter ones (DNPs); that means that in these propositions the law of double negation fails ("Two negations do not affirm") and hence these propositions do not belong to classical logic, but to intuitionist logic[9].

8) By following Leibniz's attempt of a new mechanics his theory rejects idealizations unsupported by experimental evidence and, in particular, any concept obtained by passing to the ideal limit of experimental properties of a concept (for example the perfectly hard body; truly, there exist bodies that closely approach it, but it is not realized by any real body[10]).

3. Which Is the Basic Principle of Lazare Carnot's Mechanics?

A crucial problem is the interpretation of Carnot's mechanics, i.e. to clarify the meaning of its two basic equations.

Jouguet (1908: p. 203) guessed that L. Carnot's mechanics plays a subordinate role to Lagrange's, or at most a mere preparation for Lagrange's mechanics. Ivor Grattan Guinness' appraisal of L. Carnot' 1803 book (Grattan-Guinness, 1990; section 5.2.6 and section 16.3.3 under the heading "engineering mechanics") manifests a haste in evaluating it "a mixture of Euler and Lagrange, with Euler perhaps dominant" (p. 295).

The book (Gillispie & Pisano, 2013: p. 39) looked for the basic principle of his theory. They evaluate Carnot's "first fundamental equation" (E) a "quasi tautology". Then they suggest that Lazare introduced the notion of geometrical motion in order to overcome the two different theorizations of an impact of bodies, either elastic bodies or hard bodies. By reiterating a Clifford Truesdell's suggestion for interpreting some specific developments of Newtonian mechanics (Truesdell, 1968), they considered as fundamental principle of L. Carnot's theory the conservation of the momentum of momentum (Gillispie & Pisano, 2013: pp. 23, 30-31). But no evidence is offered for this interpretation; which rather appears as an attempt to reduce Lazare's mechanics to the usual laws of Newtonian mechanics.

The preface to the Italian version of *Essai* (Drago & Manno, 1994) remarks that Carnot's formulation of the first fundamental equation (E) is inaccurate. It is derived from the principle of action and reaction (Carnot, 1783: section xi), i.e. the conservation of the momentum of an isolated system. Let us illustrate it in modern

[8]An analysis of Newton's principles through formulas of mathematical logic is in (Drago, 2017: section 2.12) sharply characterizes the differences between the idealistic Newton's versions and the experimental versions like Carnot's ones.

[9]Of course, it is a modern lecture that suggests this logical feature. However, L. Carnot was attentive to logic; he introduced a symbolic notation for the geometric magnitudes and a logical notation for the logical implication (Drago, 1989b).

[10]Actually, Carnot makes use of the concept of hard body; this notion gives troubles in the interpretation of his text; many times is equivalent to elastic body and sometimes to plastic body.

notations the equation is written by Carnot by means of the projections on a given line; here the vectorial symbols are understood: $\Sigma m_i U_i = 0$ (where m_i represents the mass of the i-th body, U_i the velocity it loses in the interaction i.e. the difference between the velocity W_i before the interaction and the velocity V_i after it). Therefore the equation is equivalent to $\Sigma m_i W_i = \Sigma m_i V_i$. In the case of hard bodies the final velocities V_i are the same for everybody; Carnot formally changes it into $\Sigma m_i U_i V_i = 0$ (or, in scalar form, $\Sigma m_i U_i V_i \cos U_i V_i = 0$) (E). This is the conservation of the momentum in the case of hard bodies, which all have a same final velocity V which is specified without necessity for each i-th body. Then, without a satisfying justification Carnot replaces V_i with the geometrical motions (which we will define later) u_i he obtains the equation:

$$\Sigma m_i u_i U_i = 0, \text{ (F).}$$

The above mentioned appraisal by Gillispe and Pisano ("a quasi tautology") is not unjustified.

Notice that the equation (E), apart the sum on the masses, is formally the same of Carnot's celebrated geometric theorem for the three sides A, B and C of a triangle: $A^2 = B^2 + C^2 + 2BC\cos bc$. Maybe he speculated on a parallelism between the two formulas; evidence for that seems the fact that in section xxiv he obtains the conservation of mechanical energy by just referring to a triangle composed by the three vectors W, V and U owing to the equation among the three velocities: $W = V + U$. There he squares it; by multiplying each term by m_i and summing up, he obtains the conservation of mechanical energy plus the term $\Sigma m_i V_i U_i$; owing to the equation (E) this term is equal to 0 and the important result, the conservation of energy is obtained (Carnot, 1783: sections xxiv-xxvi; see also Drago & Manno, 1994: fn. 18). One can suspect that in founding his theory Lazare Carnot wanted to proceed in the reverse order of this derivation; in a first time he wanted to independently justify the equation (E) and then to derive from it the conservation of energy. For this purpose he accommodates the equation (E) by attributing to U a useless index U_i: which however prepares its subsequent replacement by a geometrical motion u_i. However, his justification of this logical step is insufficient and the entire Carnot's argument is confusing (Drago & Manno, 1994: fn.s 21 and 22). In (Carnot, 1803a: p. 104) the treatment of the question is the same without adding any clarity.

Surely these shortcomings harmed the reception of his two books by contemporary scientists, who were accustomed to the rigorous and elegant derivations presented by Laplace and Lagrange. However, the following scholars admired the new consequences of the second equation and intended his theory as a technical theory.

4. The Derivation of L. Carnot's Mechanics from the PVW

Yet, one has to take into account that L. Carnot himself indicated the (methodological) principle of his new theory.

In the book of 1803 he declares to be founding his theory on the principle of virtual works (**PVW**), although adapted to the specific case of bodies' impact.

> My theory could not be founded precisely on the principle of virtual velocities…, which is not applicable, without modification, to the impact of bodies. I therefore start from a principle which is different but very analogous, or rather, which is this same principle of virtual velocities, but suitably extended (Carnot, 1803a: p. x)[11]

Indeed, in the book (Carnot, 1783: sections II and III) he deals with the so-called Torricelli's principle (i.e. a first, reduced version of the PVW) as a principle "joining a rigorous proof with a sufficient generality", so much to be "sufficient to reduce all questions to an affair of calculation and geometry, that is the true subject of mechanics". He proves the validity of this principle through an *ad absurdum* argument and concludes that it is true "without any exception". Subsequently (Carnot, 1783: section xxx) he translates the equations (F) into the expression $\Sigma F_i u_i \cos Z_i = 0$ as a (i.e. he changes $m_i u_i$ in F_i, where F_i is the force without the divisor which is common to all forces, Δt). Then in section xxxiv he reiterates the solution of the general problem of the theory in terms of "moments of activity" (i.e. elementary works): $\Sigma F_i u_i\, dt \cos Z_i = 0$: this formula is similar to PVW where the geometrical motions replace the virtual displacements.

Why this path of argumentation? In this way he started from the discrete for then arriving to a similar formula to PVW in the continuum. Moreover he goes from the momentums to the "forces" which now have nothing of metaphysical. Furthermore he arrives to obtain a similar formula to PVW. The path is not entirely formal but the results are substantial.

The subsequent theoretical development of Carnot's theory replaces the differential equations of Newton's mechanics with a different mathematical calculation. This new technique which is based on the concept of "geometric motion". Its clearest definition is given in (Carnot, 1783: section xvi): "a motion assigned to a system of bodies is geometric if it is such that the opposite movement is also possible". The introduction of this concept allows Carnot to establish the second fundamental equation, to be applied to a system of interacting bodies, as a body's motion non interacting with all bodies composing the geometric configuration of the given system (Carnot, 1803a: section 136).

But Carnot's equation is based on geometric motions whereas PVW is based on virtual displacements. The similarities and differences between the notion of geometrical motion and the modern notion of virtual displacement have been analyzed in detail. The main result is that geometric motions differ from virtual velocities in being finite, and in being possible or actual displacements, without internal work and energy consumption within the system (Drago & Manno, 1994; Gillispie & Pisano, 2013: p. 22).

Now let us demonstrate, in the manner of the scientists of the 1700s who considered all functions were as continuous curves and hence $\Delta x \approx dx$; this is the main

[11]The above mentioned scholars disregarded these L. Carnot's explicit declarations on the foundation of his theory, by evaluating them as the presentation of a specific "idiom" (Gillispie & Pisano, 2013: p. 21).

hypothesis of the following comparison, that Carnot equation (F) applied to a system of particles derives from the PVW. Let us recall textbooks' definition of this principle: *A system of particles is in equilibrium if and only if, the total virtual work of active forces is zero*; *that is, if and only if*

$$\sum F_i^a \delta s_i = 0 \tag{1}$$

But let us recall that for invertible motions the work of the constraints is null (otherwise there would be free work; hence, it expresses mathematically the impossibility of perpetual motion):

$$\sum F_i^c \delta s_i = 0\,. \tag{2}$$

Hence: $\sum F_i^a \delta s_i + \sum F_i^c \delta s_i = \sum F_i \delta s_i = \sum m_i dV_i / dt \delta s_i = 0$, i.e. $\sum Fu \cos X = 0$.

Let us go back to (2). By identifying, as was usual in the 1700s, Δ with δ (which is correct if the constraints are fixed and the forces are continuous), we have:

$$\sum m_i \Delta v_i \Delta s_i / \Delta t = \sum m_i U_i u_i \tag{3}$$

where the virtual displacement, now expressed as the motion $\Delta s/\Delta t$, is assumed to be invertible.

Now let us proceed to obtain the second fundamental equation of Lazare Carnot. If we consider together the two following relations: $m_i U_i = m_i \Delta v_i$ and $U_i = \Delta s_i / \Delta t$ (where Δs_i is the geometric displacement of the i-th point and Δt has to be specified), from (3) we obtain:

$$\sum m_i \Delta v_i \Delta s_i / \Delta t = 0 \quad \text{i.e.} \quad \sum m_i \Delta v_i / \Delta t \Delta s_i = 0\,.$$

Switching to infinitesimals we have:

$$\sum F_i \delta s_i = 0 \tag{4}$$

This equation differ from (1) each other because:

1) In (4) there is a geometric motion which gives a ds and not a virtual displacement δs_i,

2) F_i is the resultant force acting on the i-th particle, not its active component F_i^a.

However, notice that if a geometric displacement equals an invertible virtual displacement, the virtual work of the constraining force components is null. Furthermore, when constraints are present, the resulting force acting on the i-th particle is given by $F_i = F_i^a + F_i^v$, where the second term represents the constraining force acting on the i-th particle. Moreover, in applying the PVW, it is assumed that if every displacement is invertible (precisely the condition that derives from being geometric motion) then the virtual work of the constraining forces is null: $\sum F_i^c \delta s_i = 0$.

Thus, by removing the latter equation from the Equation (4) it becomes the PVW expression (1). The agreement is surprising, for we started from concepts that are quite different from the usual ones. In conclusion, through a comparison in modern terms we have established a correspondence between geometric motions and virtual displacements; it is restricted by some limitations which however allow to qualify them as similar notions.

Hence, geometrical motions surely originate from the PVW, the principle on which Lazare himself declares that his mechanics depends. Both Carnot's definition and applications of them do not have relation with the nature of the bodies, either hard or soft.

This new foundation of mechanics is important also because characterizes one more line of historical development of theoretical physics: to the line of Newton's $f = ma$ one has to add the line of the PVW (Drago, 1993; Gillispie & Pisano, 2013: Table 6.2).

Notice that, by ignored this formula, Leibniz, unsuccessfully tried to build an alternative foundation of mechanics to Newton's. L. Carnot's mechanics, based on the mathematical formula of PVW, achieved this result.

Let us now compare L. Carnot's mechanics with Mach's work book (Mach, 1893) that is widely known as a radical criticism of Newton's theory and possibly an alternative to it. Only the former one deserves the qualification of the true alternative to Newton's theory, because Mach's mechanics defines mass by voiding the third principle of physical content, lacks of the notion of system of reference and overall maintains unaltered Newtons's the second principle.[12]

5. The Past Great Debate on the Nature of PVW[13]

PVW is not an abstract idea; it is derived from another methodological principle: the impossibility of a perpetual motion. PVW is a principle because attributes, without any experimental evidence, a null work to the constraints' actions of a system solicited by external forces; because, it is argued, if constraints' work was positive, it would generate a perpetual motion; which is an impossible event.

Let us briefly consider its history. In a letter to Varignon of the year 1715, yet published in 1725, Johann Bernoulli suggested a mathematical formula for this principle. After half a century Lazare Carnot and Lagrange proposed this principle as the basis of a new way of founding Mechanics. Indeed, As Lagrange (1788: pt. II, 1.4) stressed it, the PVW is successful in theorizing the study-cases which are intractable by f=ma principle of Newton's mechanics owing to our *a priori* ignorance of constraints' reactions and hence of what has to be put as f in Newton's equation[14]. Hence, PVW constitutes a methodological principle for solving a basic

[12]Notice that Mach considered as the more advanced theory one in the Newtonian line of development Lagrange's one, without attributing to the PVW any foundational role.

[13]For an extensive history of the PVW see (Capecchi, 2012). Attention is devoted to Lazare Carnot by the book (Oliveira, 2014), which however in the history of mechanics pursues a more restricted focus, the history of the concept of work.

[14]In a letter of the year 1759 Lagrange (1892: p. 173) announced to Euler that he had arrived at treating the "true metaphysics" of the principles of mechanics, which therefore can no longer be identified with, rather with the PVW. "I have myself composed some elements of mechanics and of differential and integral calculus for the use of my students, and I believe I have developed *the true metaphysics of their principles*, as much as this is possible." Remarkably, Fraser (1983: pp. 233-4) convincingly suggested that this "true metaphysics" is merely the PVW although he did not attribute to it an alternative role to Newton's principles.

problem: how to overcome the ignorance of constraints' reactions.

Owing to Lagrange's remark PVW cannot be the consequence of Newton's principles; as a consequence of this remark and its being the origin of two essentially different theories from Newton's one, i.e. Lazare Carnot's one and Lagrange's one and also the inconclusiveness of the past debates on it (Drago, 1993), PVW is *independent* from the Newtonian theoretical framework.

For the first time in an experimental science already well-constituted in a systematic way by Newton and Euler, a new foundation born without appealing to metaphysics (as Newton had did when appealed to absolute space, absolute time and force-cause; and Maupertuis had did when he suggested the new principle of least action).

However, for the theoretical attitude of the physicists of that time, who had internalized the deductive theoretical organization as the only possible organization, the replacement of Newton's principles led them to rework the new theory at a level far higher than single laws, i.e. the level of axioms of possibly metaphysical nature. Since in the history of science the PVW instead had originated in the ongoing practice of artisans, this principle seemed of too humble origins to be comparable with the previous high level principles of Newton and Maupertuis. It was therefore maintained that it was necessary "to prove it", that is to derive it from some other elevated principles similar to that of Maupertuis or the principle of levers, which at that time enjoyed a metaphysical meaning. Unfortunately, also Lagrange misinterpreted the role played by the PVW within a theoretical organization; he wanted to prove it in order to reduce it to a consequence of more respectable principles and offered a "proof".

All the great scientists of the time engaged themselves to this undertaking: Laplace, Poisson, Poinsot, Fourier, Navier (Poinsot, 1975). Leaving aside the metaphysical demonstrations, the other proofs tried to reduce the new principle to its simplest case, e.g. the sum of forces (i.e. the then almost metaphysical notion which in subsequent times was reduced to the mathematical law of the parallelogram) or the very ancient law of the lever[15].

Each of the above mentioned demonstrations depended on a schematization of the machines which at that time was common: they were treated theoretically as if their elements had no mass (as today we consider wire in mathematical problem on pendulum)[16]. It was therefore believed that, by adding all the forces F_i of a

[15]This "proof" It was essentially considered as a question of reducing the *n* variables of the principle to just two or even one variable. It was believed that this goal could be easily achieved by exploiting the geometrical innovation of the time. A few years earlier Lorenzo Mascheroni (1797) had shown that geometric constructions which are performed by means of ruler and compass (a pair of instruments that in the ancient times had religious and philosophical significance) may be performed by means of the compass alone. Since then, theoretical importance was given to geometric instruments which until then had not been used or had been used only with distrust, that is, both linkages (rigid hinged rods, e.g. the pantograph) and tracer wheels; all gave more results than those obtained with ruler and compass. This novelty led to hope that geometry could suggest much more innovations.

[16]It was Lazare Carnot who established that machines have instead to be treated as having mass (Carnot, 1783: section XXX); however, he was not immediately listened to.

mechanical system, to which is applied the PVW, the remaining problem was only a geometrical one; to solve it no attention was devoted to the fact that the single displacement δs_i cannot be related to the total force $\Sigma_i F_i$ as unsuccessfully tried to do all "provers" of PVW. In conclusion, all these demonstrations were vicious circles.

Along decades, scientists suggested proofs that were not entirely conclusive. At last, as Poinsot testifies, the problem was abandoned due to exhaustion (Poinsot, 1975). Only few scientists had the idea that this principle didn't need to be proved, but rather it has to put as a new foundation; as actually Lagrange had done with his equation or L. Carnot with his second fundamental equation.

UnfortunatelyL. Carnot' explicit declarations on his theoretical approach, which is based on the PVW considered as independent from Newton's mechanics, have been ignored in the last two centuries. Also Lagrange's remark on the greater power of the PVW with respect to Newtonian principles was ignored long two centuries. Lagrange's formulation of mechanics was considered as no more than an elegant, more simple mathematical technique for dealing with cumbersome mechanical problems. Even Ernst Mach, although cleverly contesting the background and the notions of Newton's mechanics, agreed with this interpretation of Lagrange's theory (Mach, 1893: chap. IV, section III)[17].

6. The Second Main Innovation of Carnot's Mechanics; Its Basic Choices

The independence of PVW from $F = ma$ in founding mechanics can be proved also by means of foundational arguments.

First, let us consider Lazare's kind of mathematics.

Notice that the main axiom of Newton's mechanics, $f = ma$, requires AI mathematics because it is translated into a differential equation whose solutions may be not constructive (Drago, 1986). In addition, let us consider the variational foundations of mechanics. The variations give as solution a particular function through equality to 0. Since in constructive mathematics even in the case of real

[17]At the same time of this debate the main problem of the geometers investigating the foundations of Mathematics was to find out a proof of Euclid's fifth postulate. After an inconclusive debate, in the '30s two geometers, both living in peripheral areas with respect to the European scientific centres (the Russian Nicolai Lobachevsky and the Hungarian Janosc Bolyai) bravely suggested ~~a~~ new answers; i.e. no longer to search a theorem, rather to assume the *independence* of this postulate from the others; by looking for new versions of this postulate they developed new geometrical theories, respectively the hyperbolic geometry (based on the hypothesis of two parallel lines) and the absolute geometry (without any hypothesis on parallel lines) (Kline, 1972; Bazhanov & Drago, 2010; Drago, 2002). As a fact their theories were not axiomatic but based on a problem (of course, how much are the parallel lines). However, the births of the new theories of geometry have been long time ignored. Notwithstanding this period of misrecognition, a revolution in the foundations of geometry and more in general of the entire mathematics began (Drago, 2011). Instead in Physics is surprising that some decades before and in a parallel way of the above two mathematicians, L. Carnot and Lagrange the *independence* of the PVW from the Newtonian principles was ignored. The inconclusiveness of the debate left open the problem of the foundations of PVW and at last of the entire problem of the foundation of mechanics remained unresolved.

numbers the exact equality to 0 is undecidible, also the problem of the equality of a functional to 0 is undecidable.

Against the traditional mathematics including idealistic notions, e.g. the infinitesimals based on AI, Lazare Carnot declared the "empirical" nature of *mathematics*: "all [mathematical] ideas come from the senses" (Carnot, 1803a: pp. 2-3). Provided that the adjective "empirical" is intended as "operative" and hence, in mathematics, "algorithmic", the previous adjective qualifies a specific kind of mathematics based on PI only; at present time it is called constructive mathematics (Bishop, 1967).

PVW does not make use of infinitesimal analysis and hence AI mathematics. From the point of view of constructive mathematics the PVW is represented by a formula that poses no problems, because it is a formula of elementary mathematics and its equality is to be understood as approximate equality, as it is for all experimental laws. In addition, the displacements of his mathematical formula may be also finite displacements. There may exist problems in solving the constraint equations, to the extent that a solution of them without approximations and they may give double solutions; however, these cases seem extraneous to the usual applications.

By disregarding positions or trajectories whose determination usually requires infinitesimal analysis, Carnot's mechanics is of an only algebraic-trigonometric nature, and the solutions of the basic problem are expressed by velocities. That means that this foundations of mechanics does not appeal to idealist notions; in other words, this mechanics is entirely constructive, i.e. is based on a mathematics including only PI (D'Avino & Drago, 1983)[18].

In addition, Lazare illustrated and discussed an option concerning the two kinds of *theoretical organization* (Carnot, 1783: pp. 101-103; Carnot, 1803a: pp. 3-4).

> Among the philosophers who are occupied with research into the laws of motion, some make of mechanics an experimental science, the others, a purely rational science; that is to say, the former in comparing the phenomena of nature break them up […] in order to know what they have in common, and thereby reduce them to a small number of principal facts […]; the others begin with hypotheses, then reasoning in consequence upon their suppositions, come to discover the laws which bodies follow in their motions, if these hypotheses conform to nature, then comparing the results with the phenomena, and finding them to be in accord, conclude from this that their hy-

[18]Through a celebrated book Lazare suggested a new foundation of differential calculus (Carnot, 1813). He based it on operative calculations instead of the "chimerical beings" of infinitesimal analysis. His specific book was celebrated for having reduced the infinitesimal analysis to an operative mathematical calculus, i.e. a mathematics based on PI. He suggested the same for geometry; i.e. he suggested effective calculations for replacing what he considered idealistic notions, the isolated negative numbers (Carnot, 1803b; Boiano & Drago, 2002).

pothesis is exact. (Carnot, 1783: p. 102)[19]

He chose the theoretical organization which is alternative to Newton's deductive-axiomatic (derived from $f = ma$); it is based on a search for solving a general problem (PO): "the reduction of the phenomena at issue to a little number of main facts." (Carnot, 1783: p. 101; Carnot, 1803a: pp. 3-4) More precisely, in theoretical mechanics he states the following basic "Problem": "… given the virtual motion of any system of hard bodies… find the real motion it will assume in the next instant." In section x and also in section xx the problem is reiterated; the solution is given at the end of section xvi and with the theorem of section xxii (Carnot, 1783).

Already in the above it was underlined that L. Carnot indicated the PVW as the (methodological) principle of his theory based. This principle actually addresses to the search of a solution of the problem of the constraints' reactions (PO)[20]. This problem is solved through the PVW which suggests how to compute the evolution of a mechanical system by ignoring constraints' forces. The solution is the equation (F).

In conclusion, the relationship between physics and mathematics was reduced by L. Carnot to elementary mathematics appealing to only PI; moreover, his alternative theoretical organization was aimed at looking for the solution of a problem (PO).

Hence Carnot's mechanics is a theory whose basic choices are PI and PO[21].

All that in the above supports with great evidence the previous suggestion: the history of mechanics presents more than one line of historical development; there are essentially two lines, one of the Newtonian paradigm (plus some partially deviating theories), another of the theories based on the PVW.

Lazare illustrated only wordily both his alternative organisation to the axiomatic theory (Carnot, 1783: pp. 101-103) and his "empirical" mathematics (Carnot, 1803a: p. 3); yet, he did not theorize on them nor link these two choices–PVW this principle as the basis of his PO mechanics and the elementary mathematics PI. Therefore, these choices remained uncorrelated in the mind of the reader and maybe each of them did not appeared convincing. As a fact, they have been disregarded as specific innovations of a TP of an apparently technical

[19]An antecedent declaration on the two kinds of organization of a scientific theory was suggested by (D'Alembert, 1797: p. 501). (Frisch, 2006) illustrated the proposals for a new theoretical organization by Anton Lorenz, Henri Poincaré and Albert Einstein, who however did not suggest a detailed model of it. Cellucci (2025: chapt.s 3-5) auspicated a new theoretical organization, yet without identifying it in some instances of the past history of science. Recently Oliveira (2025: pp. 9-10) suggested the "observational-inductive model (OIM)" as "complementary" to the deductive one; yet, he does not apply this notion to specific cases.

[20]Previous works characterized the ideal model of a PO theory (Drago, 2012; Gillispie & Pisano, 2013: p. 136).

[21]A comparison of Lazare Carnot mechanics with the contemporary Lagrange's mechanics enlightened the similarities (mainly the common base, i.e. the PVW) and the basic differences (mainly the kind of mathematics; it is based on elementary mathematics (PI) the former one, on the mathematics of the infinitesimals (AI) the latter one; Capriglione & Drago 1998).

nature[22].

It is clear that each of the two above dichotomies represents a sharp and exclusive alternative. On one hand no connection between classical logic and on the other hand intuitionist logic is possible, owing to the failure of the classical law of double negation within the latter one. Moreover, a decidable problem cannot exist in a theory making use of AI, because this kind of theory overcomes any difficult theoretical problem by a suitable hypothesis of idealistic nature. Vice versa, no constructive technique can introduce the infinitesimals. Hence, two theories having different basic choices, i.e. belonging to two different MSTs, may be called incommensurable.

7. The Most Disregarded Innovation of Carnot's Mechanics: Symmetries

The mathematical techniques exploiting the geometrical motions find out the "invariants" of the impact of bodies (Carnot, 1783: section xxii). He proudly claimed to have inaugurated "a new science, intermediate between geometry and mechanics".

> 145. The theory of geometric motion is very important; it is, as I have already observed elsewhere (*Geometry of Position*, [1803b] p. 357), a kind of intermediate science between ordinary geometry and mechanics. It is the theory of the motions that any system of bodies can take without them hindering each other, without them exerting any action or reaction whatsoever on one another. This science has never been treated specifically: it is entirely yet to be created, and deserves, both for its inherent beauty and its utility, the full attention of scholars; for the great analytical difficulties encountered in analytical mechanics and especially in hydraulics arise solely from the fact that the theory of geometric motions has not yet been developed. I will confine myself here to examining the principal properties of these motions, insofar as they are necessary for the work I have undertaken. (Carnot, 1803a: p. 116)

The crux of his theory is the equation (F): $\Sigma m_i u_i U_i = 0$. The velocity u_i is an indeterminate magnitude and any specification of it gives rise, through this relation, to an equation that is applicable to the system.

For example, if we assign to the u_i the same value $u = \text{const}$ (a uniform translation of an entire system of bodies is certainly a geometric motion) we have: $\Sigma m_i u U_i = 0 \Rightarrow u\Sigma m_i U_i = 0$. From this, because u is arbitrary, it follows that: $\Sigma m_i U_i = 0$, i.e. $\Sigma m_i W_i = \Sigma m_i V_i$. This is precisely the conservation of the total momentum of the system (Carnot, 1783: section xxi).

Then we can assign another geometric motion, consisting in rotating the whole system around a fixed axis with angular velocity ω; hence $u = \omega x r_i$. (where x is the

[22]Notice that this foundation has generalised also D'Alembert's attempt to give a new foundation to mechanics (Hankins, 1970: pp. 174-176). As already many scholars concluded, D'Alembert principle is nothing else the principle of virtual works (Drago, 2000).

symbol of vectorial product). In this case we have: $\Sigma m_i\, \omega x r_i \cdot U_i = 0$. By the property of the mixed product, we have: $\Sigma m_i\ \omega \cdot r_i x U_i = 0$, hence $\omega \cdot \Sigma m_i\ r_i x U_i = 0$. By the arbitrariness of ω we have: $\Sigma m_i\ r_i x U_i = 0$. Finally, since $U = W - V$, we have: $\Sigma m_i\ r_i x W_i = \Sigma m_i\ r_i x V_i$. This last equation expresses the conservation of the moment of momentum (Carnot, 1783: section xxii).

We note that Hermann Weyl defined a thing symmetrical if it can be subjected to certain operations and it remains exactly the same as before (Weyl, 1952: p. 45). In the light of this definition it can be said that under the transformation given by a geometric motion, the result of (F) is the same; hence it manifests a symmetry.

We can easily verify that each examined case of geometric motions forms a group of transformations: clearly the translations are associative, commutative, and, inasmuch as they are by definition reversible, possess the inverse transformation. In (Carnot, 1803b) a geometric motion is defined as a motion in the absence of interaction; in this case Theorem II establishes the inverse[23]; Theorem III establishes the composition of two geometric motions. These properties all are necessary to define a group. The same holds true for the transformations of rotation.

We can therefore say that Carnot rigorously defined a group through all its specific features; moreover his technique leads the subgroup of uniform translational motions to establish the conservation of momentum; while leads the subgroup of uniform rotatory motions to the conservation of moment of momentum (We know that conservation of energy is the invariant with respect to temporal translations which are not included in (F). Indeed we saw in section 3 that Carnot derives it in another way, by exploiting his fundamental equations (E)). We conclude that Carnot essentially founded his Mechanics on the invariants of the motion provided by the application of the geometric group of spatial symmetries[24]. Moreover the above quotation says that he was full aware of the importance of his novelty.

8. The Difficult Birth of Symmetries in Theoretical Physics

The attribution of the symmetries technique to Lazare Carnot changes the traditional history of the introduction of symmetries in science. Carnot's introduction of symmetry in mechanics occurred one year before Haüy's introduction of symmetry in crystallography (Haüy, 1784). Moreover, Haüy's introduced a mere method, not a mathematical technique: he conceived a "molecule intégrante" as

[23]Felix Klein defined for the first time the inverse of a group of transformations in the year 1991, in the Italian translation by U. Fano of his celebrated paper (Erlanger program) identifying geometries with particular groups of transformations (Klein, 1890).

[24]The paper (Drago, 1989a) recognized that as first Lazare Carnot introduced the use of symmetries in theoretical physics through an algebraic-trigonometric technique treating geometrical motions. In addition, the paper (Drago, 1996) showed that symmetries are mathematical instrument of all theories relying on the choices PI and PO, as Carnot's mechanics does. Gillispie (1971) and (Gillispie & Pisano, 2013: p. 31) recognized this new mathematical technique. But they did not link this technique to the modern use of symmetries in theoretical physics.

the last element of the decomposition of crystals; he assumed that conversely a crystal is composed by the addition of a sequence of such molecules, which, owing to their form, are obliged to occupy determined locations on the crystal; whose global form constitutes the invariant of the process of addition; his law of rational indices states mere proportions between rational numbers.

50 years after Carnot, Evariste Galois suggested in algebra the notion of symmetry as due to a group of transformations. His attitude with respect to the calculus was the same of Carnot in foundational problems: no calculus (called at this time also "algebra") rather to "do the analysis of the analysis" (Galois, 1982: p. 11). Galois reiterated Carnot's method: it is based on "an adjunction" to a given system—a geometrical motion in the case of Carnot, a radical in the case of Galois. Then one easily finds out the solution which later may be emended from the adjunction, so to coming back with the solution without the adjunction.

However, whereas Carnot made use of continuous groups, the spatial ones, Galois made use of the substitution group (permutation group), i.e. the simplest discrete group. Moreover, Galois seldom made use of the word "group" and not always appropriately; instead Carnot was proud to have conceived a new theory. Furthermore, Galois did not conceive the basic property of a group, i.e. the existence of the inverse of each transformation (truly, the inverse it is trivial for the permutation group); whereas Carnot defined the inverse motion as a basic property of all geometrical motion (Drago, 2014). A long period of time elapsed before the Galois' innovations (about the simple permutation group) have been recognized (more than thirty years); this fact shows how far the scientific attitude of the scientists of that time was from symmetries (Drago, 2022).

In 1894 Pierre Curie (Curie, 1894) introduced in theoretical physics the innovations on symmetries that crystallography had obtained along the previous century. Actually, he dealt with symmetry breaking, a higher subject of the symmetry technique, illustrated his innovations by mixing the metaphysics of the causes of symmetry-and methodological suggestions. Owing to this theoretical mixture his paper resulted obscure along one century and more[25].

In 1905 Einstein radically changed mechanics by applying Lorenz group of transformation to it. Actually, he did not refer to this group if not implicitly and even the word "group" was used by him only one time for intending a mere collection[26]. He disregarded to establish the basic properties defining a set of mathematical objects as a group.

[25]Some explanations have been given by the paper (Drago, 2024).

[26]After this paper, cancelling the notions of absolute space and time, theoretical physicists had to baptize the group of transformations of classical mechanics. They called it "Galileian group", although Galilei did not gave particular attention to the velocity transformations, through the well-known instance of a ship moving with constant motion with respect to the river. Before him other antecedent scholars (Nicolaus Cusanus, Giordano Bruno, etc.) considered this instance. Rather, Christian Huygens and Leibniz attributed much importance to the relativity to velocity transformations; in particular Leibniz studied also the invariance to rotation transformations. In this light the velocity transformations would be called Leibniz's transformations. But probably Leibniz's fame to be a metaphysical philosopher obstructed this attribution.

In 1928 Hermann Weyl introduced quantum mechanics as based on group theory; but his book was de-evaluated as a merely pedagogical book. At last in 1956 Yang and Lee's discovery of the non conservation of parity in the theory of elementary particles persuaded all physicists of the relevance of symmetries.

In sum, Carnot introduced the symmetries 50 years before Galois and 100 years before Klein's Erlanger program, 120 years before Einstein's special relativity, 150 before Weyl's quantum mechanics and 170 years before Yang and Lee's discovery. Unfortunately no one of the previous list knew the antecedent discoverers. The ignorance of Carnot's priority plus the obscurity of Galois', Curie's and Einstein's introductions of groups made difficult the recognition of the great importance of group theory in science.

9. Open Problems in View of a Rational Re-Formulation of Carnot's Mechanics

First open problem: to produce a critical edition of (Carnot, 1873) in order to accurately describe: its theoretical basis, the best derivation of equation (F) obtaining the invariants, and the importance of its results. A first attempt was (Drago & Manno, 1994).

Second. L. Carnot started part of his theory from a tautology: in stating equation (E) he assumed the conservation of momentum for then obtaining the same equation. A rational re-formulation of Lazare Carnot's mechanics according to both the PVW and the alternative choices to the Newtonian ones is a need. A first answer was the paper (Bellini, Drago, & Mauriello, 2007) suggesting a scheme of the correct development of L. Carnot's mechanics.

Third. To discover a mathematical linkage between L. Carnot's formulation of mechanics and the following Hamilton's one, whose algebraic approach at present time has replaced in theoretical physics the analytical approach of the Newtonian mechanics. The basic content of Hamilton mechanics is the conservation of energy, but its origin was from optics. The aim of this research is to investigate on a possible continuity of the algebraic approach in the history of mechanics. If successful, this linkage would give a retrospective view of Carnot's mechanics as an anticipation of Hamilton mechanics; under this light new features of the former mechanics may be obtained, in particular by comparing the respective methods for obtaining symmetries.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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